TY BSC

Real Analysis & Multivariables Calculus-II

3/4/13

QP Code : AY-6685

($2\frac{1}{2}$ Hours) [Total Marks: 60

N.B.: (1) **All questions are compulsory**.

(2) Figures to the right indicate marks for respective subquestions.

1. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. If $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim\limits_{x\to a} f(x, y)$, $\lim\limits_{y\to b} f(x, y)$ both exist. Prove that

$$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right] = L.$$

Give an example to show that the converse is not true. (7)

(b) Attempt any two questions:

(i) State and prove Mean Value theorem for scalar fields (4)

(ii) Let $f(x,y) = \begin{cases} 0 & \text{if } y \le 0 \text{or if } y \ge x^2 \\ 1 & \text{if } 0 < y < x^2 \end{cases}$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ along $y = mx$ for any m. Is $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$? Justify your answer. (4)

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |x| + |y|$. Check whether (4)

(1) $D_u f(0, 0)$ exists for an arbitrary unit vector u

(2) f is continuous at $(0, 0)$.

(iv) Determine whether the partial derivatives of f exist at $(0, 0)$ for $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x, y) \cdot T(x, y)$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. (4)

2. (a) Let S be an open subset of $\mathbb{R}^2$ and $f : S \to \mathbb{R}$ be such that $D_1 f$, $D_2 f$, $D_{12} f$, $D_{21} f$ exist on S. If $(a, b) \in S$ and $D_{12} f$, $D_{21} f$ are continuous on S, then show that (7)

$$D_{12} f(a, b) = D_{21} f(a, b).$$

(b) Attempt any two questions:

(i) If $f, g : S \to \mathbb{R}$ are differentiable on S, where S is an open subset of $\mathbb{R}^n$, then show that

(1) $\nabla(fg) = f\nabla g + g\nabla f$

(2) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ at points where $g \neq 0$. (4)

(ii) Find the direction derivatives of $f(x, y, z) = 3x - 5y + 2z$ at $(2, 2, 1)$ in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$. (4)

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is differentiable at $(0, 0)$. (4)

(iv) Determine the second order Taylor formula $f(x, y) = e^x \cos y$ at $x = 0$, $y = \frac{\pi}{2}$. (4)

3. (a) For the surface $\overline{r}(u, v)$ described by the vector equation $\overline{r}(u, v) = X(u, v)\hat{i} + Y(u, v)\hat{j} + Z(u, v)\hat{k}$, $(u, v) \in T$ where X, Y, Z are differentiable on T, define the fundamental vector product $\dfrac{\partial \overline{r}}{\partial u} \times \dfrac{\partial \overline{r}}{\partial v}$. If C is a smooth curve lying on the surface, then show that $\dfrac{\partial \overline{r}}{\partial u} \times \dfrac{\partial \overline{r}}{\partial v}$ is normal to C at each point. (7)

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(b) Attempt any two questions:

(i) Show that the area of the surface of surface of revolution of the curve $z = f(x)$ for $x \in [a, b]$ around x-axis is given by (4)

$$A(S) = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}dx$$

(ii) Find the equation of the tangent plane to the given parametric surface $\overline{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ for $x = u + v$, $y = u \cos v$, $z = u \sin v$ at $(1, 1, 0)$. (4)

(iii) Use Stoke's theorem to calculate

$$\iint_S \text{curl}\ \overline{F} \cdot \widehat{n} dS,$$

where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1$, $x \geq 0$ cut by the cone $x^2 = y^2 + z^2$ for $\overline{F} = x^2y\widehat{i} + z\widehat{j} + y\widehat{k}$. (4)

(iv) Assuming S and V satisfy the conditions of the Divergence Theorem, with usual notations, prove that (4)

(1) $|V| = \frac{1}{3}\iint_V \overline{r} \cdot \widehat{n} dS$ where $\overline{r} = x\widehat{i} + y\widehat{j} + z\widehat{k}$ and $|V| =$ volume of V.

(2) $\iint_S \text{curl}\overline{F} \cdot \widehat{n} dS = 0$.

4. Attempt any three questions:

(a) Find a constant c so that at any point of intersection of the two spheres $(x - c)^2 + y^2 + z^2 = 3$ and $x^2 + (y - 1)^2 + z^2 = 1$, the corresponding tangent planes are perpendicular to each other. (5)

(b) Compute the matrices $Dg(1, 1)$, $D(f(g(1, 1)))$ and $D(f \circ g(1, 1))$ and verify that

$$D(f \circ g(1, 1) = D(f(g(1, 1)))Dg(1, 1)$$

for $f(u, v) = (uv, u^2 + v^2)$, $g(x, y) = (x + y, x - y)$. (5)

(c) Let $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. Find $\lim\limits_{x \to 0}\left(\lim\limits_{y \to 0} f(x, y)\right)$ and $\lim\limits_{y \to 0}\left(\lim\limits_{x \to 0} f(x, y)\right)$. Does $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ exist? Justify. (5)

(d) Verify Stoke's Theorem for $\overline{F}(x, y, z) = 3y\widehat{i} + 4z\widehat{j} - 6x\widehat{k}$, S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane oriented upwards. (5)
