Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix $A = [a_{ij}]$. Then show that the entry $a_{ij}^{(k)}$ in i^{th} row and j^{th} column of A^k is the number of distinct $v_i - v_j$ walks of length k in G.
   ii. State and prove $Havel - Hakimi$ theorem for degree sequence of a graph G.

   (b) Attempt any **TWO** questions: (12)
   i. Define isomorphism of graphs. Give an example of non isomorphic graphs that has equal number of vertices and equal number of edges. Justify your answer.
   ii. Show that in a party of 6 or more people, either there are 3 persons who know one another or there are three persons who do not know one another.
   iii. If G is simple graph with p vertices, q edges and k components, then prove that $q \geq p - k$.

2. (a) Attempt any **ONE** question: (8)
   i. State and prove Cayley's formula for spanning trees.
   ii. Define Cut Edge in a simple graph G. Show that an edge e of a graph G is a cut edge if and only if there exists two vertices x and y such that e lies on every $x - y$ path in G.

   (b) Attempt any **TWO** questions: (12)
   i. Show that there exist a tree with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ if and only if 1) $d_i \geq 1$, for $1 \leq i \leq n$ and 2) $\sum_{i=1}^{n} d_i = 2n - 2$
   ii. Show that every non-trivial graph contains at least two vertices which are non-cut vertices.
   iii. State Huffman's algorithm for prefix code.

3. (a) Attempt any **ONE** question: (8)
   i. If a connected graph G contains exactly two vertices of odd degree say x and y, then show that it contains a $(x, y)-$Eulerian trail.
   ii. If G is a graph on p vertices with $p \geq 3$ such that $deg(u) + deg(v) \geq p$ for every pair of non adjacent vertices u and v in G, then prove that G is Hamiltonian.

   (b) Attempt any **TWO** questions: (12)
   i. Define closure of a graph $C(G)$ and show that a simple graph is Hamiltonian if and only if its closure is Hamiltonian.

ii. Prove that the cube graph Q_k is bipartite $k-$regular graph with 2^k vertices.

iii. Describe Fluery's Algorithm to find a closed Eulerian trail.

4. Attempt any **THREE** questions: (15)

(a) Let G be a simple graph and $\delta(G) \geq 2$, then show that there exists a cycle of length at least $\delta(G) + 1$ in G.

(b) Explain Dijkstra's algorithm and show that Dijkstra's algorithm produces the shortest path.

(c) Show that any two longest paths in a connected graph G has a vertex in common.

(d) Describe Depth First Search (DFS) algorithm. Use DFS to find spanning tree for the Complete graph K_5.

(e) Show that the line graph a simple graph G is a path if and only if G is a path.

(f) Let G be a connected graph with $2n$ odd vertices with $n \geq 1$. Show that $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots E_n$ so that $< E_i >$ is an open trail for each i.

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