Maths-III
Topology
16/10/15

QP Code : 12881

Duration: 2 $\frac{1}{2}$Hrs [Marks: 75]

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any one question: (8)
   i. Let (X, d) be a metric space. Define limit point of $F \subseteq X$. Also show that F is closed if and only if F contains all it's limit points.
   ii. In a metric space (X, d) , prove that arbitrary union of open sets is open in X. Give an example to show that arbitrary intersection of open sets is not open in X.

   (b) Attempt any Two questions: (12)
   i. Define a normed linear space $(X, \| \|)$. Show that if $(X, \| \|)$ is a normed linear space then $d : X \times X \longrightarrow \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on X
   ii. Prove that in any metric space (X, d), $A \subseteq X$, A is closed if and only if $\partial A \subseteq A$ where ∂A denotes the boundary of A.
   iii. Prove that $(\mathbb{N}, d)$ and $(\mathbb{N}, d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{N}$, are equivalent metric spaces.
   iv. Show that $A = \{x \in \mathbb{Q} : 3 < x^2 < 5\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual metric.

2. (a) Attempt any one question: (8)
   i. Let (X, d) be a metric space, $A \subseteq X$. Show that $p \in \bar{A}$ if and only if $\exists$ a sequence $(x_n) \in A$ such that (x_n) converges to p in X.
   ii. Define complete metric space. Prove that a subspace(Y, d) of a complete metric space (X, d) is complete if and only if Y is closed .

   (b) Attempt any Two questions: (12)
   i. Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.
   ii. Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. iF (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2).
   iii. Check if Cantors Theorem is applicable in the following examples. Also , find $\bigcap_{n \in \mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual.
   (I) $F_n = [n, \infty)$
   (II) $F_n = (0, \frac{1}{n})$
   iv. Let (X, d) be a metric space and (x_n) and (y_n) be Cauchy sequences in X. Show that $(d(x_n, y_n))$ is a convergent sequence in $\mathbb{R}$ (distance usual).

3. (a) Attempt any One question: (8)
   i. Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Prove that f is continuous on X if and only if for each open subset G of Y, $f^{-1}(G)$ is an open subset of X.

ii. Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Show that f is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y. (12)

(b) Attempt any Two questions:

i. Let (X, d) be a metric space and $f : X \longrightarrow \mathbb{R}$, ($\mathbb{R}$ with usual metric) is continuous on X. If $f(x_0) > 0$ for some $x_0 \in X$ then show that $\exists\ \delta > 0$ such that $f(x) > 0\ \ \forall x \in B(x_0, \delta)$.

ii. Let (X, d) and (Y, d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous on X. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

iii. Let (X, d) and (Y, d') be metric spaces. When is $f : X \longrightarrow Y$ said to be uniformly continuous? Show that $f(x) = \dfrac{1}{(1 + x^2)}$ is uniformly continuous on $\mathbb{R}$ (under usual metric).

iv. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous map. Show that $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $g(x, y) = f(x + y, x - y)$ is continuous on $\mathbb{R}^2$.(Under usual metric of $\mathbb{R}$ and Euclidean metric of $\mathbb{R}^2$)

4. Attempt any Three questions: (15)

(a) Show that $\| \ \|$ is a norm on X, where $X = M_2(\mathbb{R})$ and $\|A\| = \max_{1 \le i,j \le 2} \{|a_{ij}|\}$ for $A = (a_{ij})$

(b) Let (X, d) be a metric space and $A \subseteq X$. Show that A is open if and only if $A = A^\circ$ (Interior A). Hence decide whether the set of rationals is an open subset of $\mathbb{R}$ under usual metric of $\mathbb{R}$.

(c) Let $X = C[0, 1]$ and d_1 be the metric induced by $\| \ \|_1$ on X. ($\|f\|_1 = \int_0^1 |f(t)|dt$). Show that the following sequence of functions $\{f_n\}$ is bounded in (X, d_1)

$$f_n(t) = \begin{cases} 8n^2 t & \text{if } 0 \le t \le \frac{1}{4n} \\ -8n^2 t + 4n & \text{if } \frac{1}{4n} < t \le \frac{1}{2n} \\ 0 & \text{if } \frac{1}{2n} < t \le 1 \end{cases}$$

(d) Let (X, d) be a metric space and $A \subseteq X$. Prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

(e) Show that $f : (\mathbb{R}^2, d_1) \longrightarrow (\mathbb{R}, d)$ defined by $f(x, y) = x + y$ is continuous on $\mathbb{R}^2$, where d_1 is Euclidean metric on $\mathbb{R}^2$ and d is usual metric on $\mathbb{R}$

(f) Let (X, d) and (Y, d') be metric spaces. Show that if $f : X \longrightarrow Y$ is uniformly continuous on X and if (x_n) in X is Cauchy then show that the sequence $(f(x_n))$ is Cauchy in Y.
