Math-I
Real Analysis
15/10

QP Code : 12866

$(2\frac{1}{2})$ *Hours* **Total Marks : 75**

N.B Instructions:-

1. All questions are compulsory
2. From Question 1 , 2, and 3 Attempt any one from part (a) and two from part (b).
3. From Question 4. Attempt any three.
4. Figures to the right indicate marks.

Q.1(a) (i) State and prove First Fundamental Theorem of calculus. (8)

(ii) If f is Riemann integrable on [a,b] and a < c < b then show that f is Riemann integrable on [a,c] and [c,b] and further $\int_a^b f = \int_a^c f + \int_c^b f$. (8)

(b) (i) State and prove Mean Value Theorem for integrals. (6)

(ii) Prove that if $f:[a,b]\to\mathfrak{R}$ is Riemann integrable then $|f|$ is Riemann integrable. Is the converse true? Justify. (6)

(iii) Express the sum as a Riemann Sum of a suitable function and evaluate (6)

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{1+n^2}}+\frac{1}{\sqrt{2^2+n^2}}+\ldots+\frac{1}{\sqrt{n^2+n^2}}\right).$$

(iv) Let $f:[0,1]\to\mathfrak{R}$ defined by $f(x)=x^3$. Let $\{P_n\}$ be a sequence of partitions, given by $P_n=\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\right\}$. Calculate $U(P_n, f)$, $L(P_n, f)$ and show that $\lim_{n\to\infty} U(P_n,f)=\lim_{n\to\infty} L(P_n,f)$ and hence find $\int_0^1 f(x)dx$. (6)

Q2 (a) (i) Define double integral of a bounded function $f:E\to\mathfrak{R}$ where E=[a,b]x[c,d] is a rectangle in $\mathfrak{R}^2$. Further show with usual notations $m(d-c)(b-a)\le\iint_E f\le M(b-a)(d-c)$. (8)

(ii) State & prove Fubini's theorem for a rectangular domain in $\mathfrak{R}^2$. (8)

(b) (i) Use suitable change of variables to show that $\iint_S f(xy)dxdy=\log 2\int_1^2 f(u)du$, where S is the region in the first quadrant bounded by the curve $xy = 1$, $xy = 2$, $y = x$ & $y = 4x$. (6)

(ii) Using spherical co-ordinate, find the volume of the solid S bounded by the sphere $x^2+y^2+z^2=16$. (6)

(iii) Sketch the region and evaluate the integral by reversing the order of integration $\int_0^1\int_{\sqrt{y}}^1\sqrt{x^2+1}$. (6)

(iv) Evaluate $\iiint_S (x+y+z)dxdydz$ where S is the parallelepiped bounded by the planes x+y+z=1 & x+y+z=2 , x-y+z=2 & x-y+z=3 and x-y-z=3 & x-y-z=4. (6)

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(a) (i) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\Re$. If $\{f_n\}$ converges uniformly to a function f on S then show that f is continuous on S. Further show that (8)

$$\lim_{n\to\infty}\lim_{x\to p} f_n(x) = \lim_{x\to p}\lim_{n\to\infty} f_n(x) \text{ for each } p\in S.$$

(ii) Let $\{f_n\}$ be a sequence of Riemann integrable function on [a,b]. If the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on [a,b] then show that f is Riemann integrable on [a,b] and $\int_a^b\left(\sum_{n=1}^{\infty} f_n\right)dx = \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)dx\right)$. (8)

(b) (i) Define radius of convergence of a power series. If power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $r > 0$ then show that it converges uniformly in $[-s,s]$ for $0 \le s < r$. (6)

(ii) Show that the series $\sum x^n e^{-nx}$ converges uniformly on $[0,\infty)$. (6)

(iii) Examine whether $\int_0^1 \sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]dx = \sum_{n=1}^{\infty}\int_0^1 \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]dx$. Is the series $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ uniformly convergent on [0,1] ? Justify. (6)

(iv) Discuss the pointwise and uniform convergence of $\{f_n\}$, where $f_n(x) = n^2x^2e^{-nx}$ on $[0,\infty)$. In case $\{f_n\}$ does not converge uniformly on $[0,\infty)$, check whether it converge uniformly on [0,a] or $[a,\infty)$ where $a > 0$. (6)

Q. 4 (a) Let $f:[0,1]\to\Re$ be defined by $f(x)=\begin{cases}1-2x & \text{if } 0\le x\le\frac{1}{2}\\ 2x-1 & \text{if } \frac{1}{2}<x\le 1\end{cases}$ and $F:[0,1]\to\Re$ be defined by $F(x)=\int_0^x f(x)dx, x\in[0,1]$. Discuss (i) continuity of f at $\frac{1}{2}$ (ii) differentiability of F at $\frac{1}{2}$ when $F'\left(\frac{1}{2}\right)$ exists & check whether $F'\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)$. (5)

(b) State Riemann's criterion for integrability of a bounded function defined on [a,b] and use it to prove that the function $f(x)=x, x\in[0,1]$ is Riemann integrable. (5)

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(c) Evaluate $\iiint_E xyz\,dx\,dy\,dz$, E is the region bounded by x=0, y=0, z=0 & x+y+z = 1. (5)

(d) Find the volume bounded by the cylinders $y^2 = x$, $x^2 = y$ and the planes z = 0, x+y+z=2 (5)

(e) Discuss the pointwise & uniform convergence of the series $\sum_{n=1}^{\infty} x^n(1-x)$ on [0,1]. (5)

(f) Let $f_n : [0,1] \to \Re$ be defined by $f_n(x) = \frac{nx}{1+nx}$ for $x \in [0,1]$. Find the pointwise limit f of $\{f_n\}$. Show that $\int_0^1 f(x)dx = \lim_{n \to \infty} \int_0^1 f_n(x)dx$. Does $\{f_n\}$ converges uniformly to f on [0,1] ? Justify your answer. (5)