(3 Hours) [Total Marks : 100

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

i. Multiplicative inverse of a real number
(a) Exists and is unique
(b) Does not exist
(c) If exists then is unique
(d) None of these

ii. If $A = (2, 5]$ then
(a) $Inf\ A \in A$ (b) $Inf\ A \in A$, $sup\ A \in A$
(c) $Sup\ A \in A$ (d) None of these

iii. If $0 < x < 1$ then
(a) $x^2 > x$ (b) $x^2 > 1$
(c) $x^2 < x$ (d) None of these

iv. The sequence (x_n) where $x_n = n^3$, $\forall n \in \mathbb{N}$ is
(a) Convergent (b) Bounded
(c) Divergent (d) None of these

v. Every constant sequence in $\mathbb{R}$ is
(a) Convergent (b) Bounded but not convergent
(c) Never Cauchy (d) None of these

vi. $\lim\limits_{x \to -1} \frac{3x^2-5x-8}{x+1}$ equals
(a) -11 (b) 11
(c) 2 (d) None of these

vii. $\lim\limits_{x \to \infty} \frac{8x^2-5x+4}{4x^2+1}$ equals
(a) 2 (b) 4
(c) 0 (d) None of these

viii. If (x_n) of real numbers satisfies, $\frac{1}{n} \le x_n \le \frac{1}{\sqrt{n}}$, $\forall n \in \mathbb{N}$ then (x_n)
(a) Converges to 0 (b) Diverges
(c) Converges to 1 (d) None of these

[P.T.O.]

ix. The inequality $|x+y| \leq |x|+|y|, \forall x, y \in \mathbb{R}$ is
(a) AM-GM inequality
(b) Cauchy Schwarz inequality
(c) Triangle inequality
(d) None of these

x. The function $f(x)=e^{x}$ is continuous
(a) Only if $x>0$ (b) Only if $x<0$
(c) For each $x \in \mathbb{R}$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. State any four properties of $\mathbb{R}$ under addition. Further prove that additive inverse of a real number is unique.

ii. If $x, y \in \mathbb{R}$ such that $x<y$, then prove that there exists $r \in \mathbb{Q}$ such that $x<r<y$.

b) Attempt any TWO questions from the following: (12)

i. Prove the following: For $x \in \mathbb{R}$ and $r>0$, $|x|<r$ if and only if $-r<x<r$.

ii. Let A be any non-empty, bounded above subset of $\mathbb{R}$. Let $k>0$. Prove that $\sup(kA)=k \sup A$.

iii. Show that if $x \in \mathbb{R}$ then there exists $n \in \mathbb{N}$ such that $x<n$.

iv. State and prove Hausdorff property of $\mathbb{R}$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let (x_n) and (y_n) be two sequences converging to p and q respectively. Prove that (x_n+y_n) converges to $p+q$ and (cx_n) converges to cp where $c \in \mathbb{R}$.

ii. Prove that every Cauchy sequence of real numbers is convergent.

b) Attempt any TWO questions from the following: (12)

i. Let $x_n=b^n$, $\forall n \in \mathbb{N}$ where $0<b<1$. Show that (x_n) converges to 0.

ii. Let $x_n=3-\frac{2}{n}$, $\forall n \in \mathbb{N}$. Show that (x_n) is monotonic increasing and bounded above. Is (x_n) convergent?

iii. Prove that every convergent sequence of real numbers is bounded.

iv. Show that the sequence $\left(\cos\frac{n\pi}{2}\right)$ is divergent.

Q.4 a) Attempt any ONE question from the following: (08)

i. State and prove Sandwich theorem for limit of a function.

ii. Let $f, g: \mathbb{R} \to \mathbb{R}$ be two functions and let $a \in \mathbb{R}$. If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then prove that $\lim_{x \to a} (5f + 6g)(x) = 5l + 6m$, using $\epsilon - \delta$ definition.

b) Attempt any TWO questions from the following: (12)

i. Prove that $f(x) = 2x + 12$ is continuous at $x = 2$, using $\epsilon - \delta$ definition.

ii. Draw graph of a function $f(x) = \log_e x$ for $x \in (0, \infty)$.

iii. Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $l \in \mathbb{R}$. Give definition of $\lim_{x \to \infty} f(x) = l$ and also find $\lim_{x \to \infty} \frac{x^4 - 5}{2x^4 + 3}$.

iv. Let $f: \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Prove that $\lim_{x \to a} |f(x)| = 0$ if and only if $\lim_{x \to a} f(x) = 0$.

Q.5 Attempt any FOUR questions from the following: (20)

a) If A, B are non-empty, bounded subsets of $\mathbb{R}$, then show that the set $A \cap B$ is bounded.

b) State and prove the Arithmetic-Geometric Mean inequality for $a, b \in \mathbb{R}$.

c) Give an example of two divergent sequences (x_n) and (y_n) such that their product $(x_n y_n)$ is convergent.

d) State and prove Sandwich theorem for sequences of real numbers.

e) Discuss the continuity of the following function at $x = 4, 8$

where $f(x) = \begin{cases} 5x + 12 & if\ x < 4 \\ 3x - 2 & if\ 4 \le x < 8 \\ 2x + 6 & if\ x \ge 8 \end{cases}$

f) Prove that $f(x) = \begin{cases} -2 & if\ x \in \mathbb{Q} \\ 2 & if\ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is discontinuous at $x = 2$ by using sequential definition of continuity.

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