Q. P. Code : 51415

Duration $2\frac{1}{2}$Hrs   Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One question: (8)
   (i) State and prove Bessels Inequality.
   (ii) Let f be a continuous real valued periodic function, defined on $[-\pi, \pi]$ and having period 2π. If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series of f on $[-\pi, \pi]$ then prove that : $\sigma_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, where $\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$, S_k is the k^{th} partial sum of the Fourier series of f.

   (b) Attempt any Two questions: (12)
   (i) Is the series $\sum_{n=1}^{\infty}\left[\frac{\cos nx + \sin nx}{n^{\frac{1}{2}}}\right]$ the Fourier series of a function $f \in C[-\pi, \pi]$? Justify your answer.
   (ii) Let $f \in C[-\pi, \pi]$ and f has Fourier sereis $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, show that
   $$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)(a_k \cos kt + b_k \sin kt)$$
   (iii) Apply Parseval's equality to the function $f(x) = x$ over $[-\pi, \pi]$ and deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$
   (iv) Define Fejer's Kernel $K_n(t)$. Prove that $K_n(t) = \frac{\sin^2(\frac{nt}{2})}{2n\sin^2\frac{t}{2}}$ $-\infty < t < \infty$, $t \neq 2k\pi, k \in \mathbb{Z}, t \in \mathbb{R}$

2. (a) Attempt any One from the following: (8)
   (i) Suppose K is a compact subset of $\mathbb{R}^n$. Show that K is sequentially compact.
   (ii) Show that a compact subset of $(\mathbb{R}^n, d)$ where d Euclidean, is closed and bounded . Give an example to show that a closed and bounded subset of a metric space is not compact.

   (b) Attempt any Two from the following: (12)
   (i) Prove that a subset of a discrete metric space is compact if and only if it is finite.
   (ii) (X, d) is a metric space and (x_n) is a sequence in X such that (x_n) converges to some point $p \in X$. If $S = \{x_n : n \in \mathbb{N}\} \cup \{p\}$ then show that S is compact by using the definition of compactness.
   (iii) Let A, B be compact subsets of $(\mathbb{R}, d)$, distance d being usual. Show that $A \times B$ is a compact subset of $(\mathbb{R}^2, d')$ where d' is the Euclidean distance.

(iv) Consider the metric space $(\mathbb{R}, d)$, where d is the usual distance . Show that $\{(\frac{1}{n},\ 1) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. Is $(0, 1)$ compact ? Justify your answer.

3. (a) Attempt any One of the following: (8)

(i) Show that a subset $E \subset \mathbb{R}$ (with respect to usual metric of $\mathbb{R}$) is connected if and only if E is an interval.

(ii) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \longrightarrow \{1, -1\}$ is constant.

(b) Attempt any Two questions: (12)

(i) Prove that a metric space (X, d) is connected if and only if for each nonempty proper subset A of X, $\overline{A} \cap \overline{X \setminus A} \neq \emptyset$ i.e. $\partial A \neq \emptyset$

(ii) Let (X, d) be a discrete metric space. If A is a X having more than one element, show that A is disconnected.

(iii) If A and B are two connected subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$ then prove that $A \cup B$ is connected.

(iv) If (X, d) is a connected metric space and (Y, d') is any metric space where Y is a finite set, then show that any continuous function $f : X \longrightarrow Y$ is constant.

4. Attempt any Three of the following: (15)

(a) $f(x) = \pi + x, \ -\pi \leq x \leq \pi$. Find the Fourier series of f. Assuming that the Fourier series of f converges to $f(x)$ at $x = \dfrac{\pi}{2}$, find the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(b) If the series $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges uniformly on $[-\pi, \pi]$ to f, prove that the Fourier series of f is $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

(c) If A, B are disjoint non-empty subsets of $(\mathbb{R}^n, d)$, d being Euclidean distance and A is closed , B is compact then show that $d(A, B) > 0$.

(d) If $\{F_n\}$ is a family of closed subsets of $\mathbb{R}^n$ (distance being Euclidean) . If F_{n_0} is bounded for some n_0, and $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$ then prove that $\bigcap_{n \in \mathbb{N}} F_n$ is compact.

(e) Prove that a convex subset of a normed linear space is path connected.

(f) Prove or disprove: The subset $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ of $(\mathbb{R}^2, d)$ (d being Euclidean distance) is connected.