Q. P. Code : 51411

Duration 2 $\frac{1}{2}$Hrs Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) State and prove Cantor's Intersection Theorem for a metric space (X, d).
   (ii) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite signs then using Nested Intervals Theorem, prove that there exists $c \in (a, b)$ such that $f(c) = 0$.

   (b) Attempt any Two from the following: (12)
   (i) Let (X, d) be a complete metric space and (Y, d_Y) is a subspace of (X, d). If (Y, d_Y) is complete then show that Y is a closed subset of X.
   (ii) Prove that a finite metric space is complete.
   (iii) Show that $[0, 1]$ is uncountable.
   (iv) Prove that the set of real numbers $\mathbb{R}$ is complete with respect to the usual distance.

2. (a) Attempt any One from the following: (8)
   (i) Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Show that f is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.
   (ii) Let (X, d) and (Y, d') be metric spaces. If (X, d) is compact and $f : X \longrightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y.

   (b) Attempt any Two from the following: (12)
   (i) Let (X, d) and (Y, d) be metric spaces then show that $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(B^{\circ}) \subseteq (f^{-1}(B))^{\circ}$, for each subset B of Y.
   (ii) Let (X, d) and (Y, d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous on X. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.
   (iii) Show that the identity function $i : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d_1), i(x) = x \ \forall x \in \mathbb{R}$ is discontinuous everywhere in $\mathbb{R}$ where d is the usual distance and d_1 is the discrete metric.
   (iv) Let (X, d) and (Y, d') be metric spaces and $D \subseteq X$ be a dense subset of X. If $f : X \longrightarrow Y$ is a continuous onto map, show that $f(D)$ is dense in Y.

3. (a) Attempt any One from the following: (8)
   (i) Prove that a subset E of $\mathbb{R}$ is connected if and only if it is an interval. (Distance in $\mathbb{R}$ being usual)
   (ii) Prove that a metric space is connected if and only if every continuous function from X to $\{1, -1\}$ is a constant function.

   (b) Attempt any Two from the following: (12)
   (i) If (X, d) is a metric space and A, B are conncected subsets of X such that $A \cap B \neq \emptyset$ then prove that $A \cup B$ is a connected set.
   (ii) Prove that a convex subset of a normed linear space is path connected.

(iii) Prove or disprove: The subset $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ of $(\mathbb{R}^2, d)$ (d being Euclidean distance) is connected.

(iv) If (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ (distance in $\mathbb{Z}$ being usual distance) is a continuous function then prove that f is a constant function.

4. Attempt any Three from the following: (15)

(a) Use the intermediate value property to show that there is a square whose diagonal has length between r and $2r$ and has area equal to half the area of the circle of radius r.

(b) Check if Cantor's Theorem is applicable in the following examples. Also , find $\bigcap_{n \in \mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual.

(I) $F_n = [n, \infty)$

(II) $F_n = (0, \frac{1}{n})$

(c) If $T : \left[0, \frac{1}{3}\right] \longrightarrow \left[0, \frac{1}{3}\right]$ is defined as $T(x) = x^2$, then show that T is a contraction map on $\left[0, \frac{1}{3}\right]$. Does T have any fixed points? If yes, how many? Justify your answer.

(d) Discuss the uniform continuity of $f : [1, \infty) \longrightarrow \mathbb{R}$ (distance being usual), defined by $f(x) = \dfrac{1}{x}$.

(e) Prove or disprove: If A° and ∂A are connected then A is connected.

(f) Show that $E = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 - y^2 = 1\}$ is path connected.