QP CODE: 25880

REVISED COURSE [Max Marks:75]

Duration: $2^1/_2$ Hours

N.B. 1. All questions are compulsory.
2.From Question 1,2 and 3, Attempt any one from part(a) and any two from part(b).
3. From Question 4, Attempt any THREE
4. Figures to the right indicate marks for the respective parts.

Q.1 a i Let $\{f_n\}$ be a sequence of Riemann integrable functions on [a, b]. If the series $\sum_{n=1}^{\infty} f_n$ of functions converges uniformly to f on [a, b]. Show that f is Riemann integrable on [a, b] and $\int_a^b (\sum_{n=1}^{\infty} f_n$ (x))dx=$\sum_{n=1}^{\infty} \int_a^b f_n(x)dx$. (8)

ii Let $\{f_n\}$ be a sequence of real valued functions defined on a non-empty subset S of R. Show that $\{f_n\}$ converges uniformly to a function f if and only if for given $\epsilon > 0$ ∃ a positive integer n_o such that $|f_n(x) - f_m(x)| < \epsilon$, $\forall$ m, n $\geq$ $n_o$and $\forall x \in S$.

b i State and prove Weierstrass M - test for uniform convergence of series of functions. (12)

ii By integrating a suitable power series over an interval [0,1], show that $\frac{1}{2} = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)}$.

iii Discuss the uniform convergence of the sequence of functions $\{f_n\}$on [0, 1], where $f_n: [0,1] \to \mathbb{R}$ is defined by $f_n(x) = nx^n(1-x)$.

iv Discuss the uniform convergence of the series of functions $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$; . $x \in \mathbb{R}$

Q.2 a i If $z_0 \in \mathbb{C}$ then show that $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$. Also using definition of differentiability, show that if $f'(z_0), g'(f(z_0))$ exist then prove that the function $F(z) = g(f(z))$ has a derivative at z_0 and $F'(z_0) = g'(f(z_0))f'(z_0)$. (8)

ii $\Omega \subset \mathbb{C}$ is a domain in $\mathbb{C}$. If $u, v : \Omega \to \mathbb{R}$ are such that u_x, u_y, v_x, v_y exist ,satisfy Cauchy Riemann equations and u_x, u_y, v_x, v_y are continuous on Ω, prove that $f(z) = u(x, y) + iv\ (x, y)$ is analytic in Ω.

b i Using the definition, discuss differentiability of the function f where (12)

$$f(z) = \begin{cases} \bar{z}^2/_z, & z \neq 0 \\ 0, & z = 0 \end{cases} \quad \text{at } (0,0)$$

ii $f$is analytic on a given domain D. If $|f(z)|$ is constant on D, show that $f(z)$ must be constant throughout D.

iii Show that $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

iv Find the image of the set $|z| = 6$, $^{-\pi}/_4 \leq \arg(z) \leq \frac{3\pi}{4}$ under the reciprocal map $w = \frac{1}{z}$ in the extended complex plane . .

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Q.3 a i Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then prove that $f(z_0) = \frac{i}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}$ (8)

ii Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then prove that in the interior of the disk of convergence $S'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$.

b i If a function f is analytic at a given point then show that its derivatives of all orders are analytic at that point too. Further suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R and if M_R denotes the maximum value of $|f(z)|$ on C_R then show that $|f^n(z_0)| \leq \frac{n!M_R}{R^n}, n = 1,2,3, \ldots$ (12)

ii If z_1 is a point inside the circle of convergence $|z-z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ then show that the series must be uniformly convergent in the closed disk $|z-z_0| \leq R_1$,where $R_1 = |z_1 - z_0|$.

iii Compute the residue of $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at its simple poles.

iv State Laurent's Theorem. Expand $f(z) = \frac{1}{z(z-1)}$ as a Laurent series for the annular domains: $0 < |z-1| < 1$, $1 < |z-1|$.

Q.4 i If a real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r, then show that it converges uniformly on $[-s, s]$ where $0 \leq s < r$. (15)

ii Show that the sequence of functions $\{\frac{nx}{nx^2+1}\}$ converges uniformly on $[a, \infty)$ where $0 < a < \infty$

iii Test differentiability of the function $f(z) = z|z|$ at $(0,0)$.

iv Construct a linear fractional transformation that maps $0,\ i,\ \infty$ to $-1,\ 0,\ 1$ respectively.

v Evaluate $\int_C \frac{z-1}{(z+1)^2\,(z-2)}\,dz$, where C is the circle $|z-i| = 2$.

vi Show that $\left|\int_C \frac{e^z}{z\ \ +1}\,dz\right| \leq \frac{8\pi e^4}{3}$ where $|z| = 4$.

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