05/10/18 TYBSC

Paper / Subject Code: 10119 / Mathematics : Paper II Algebra II. (Old)

Q. P. Code: 27452

$2\frac{1}{2}$ Hours] [Total Marks: 75

N.B.: (1) **All questions are compulsory.**
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. State and prove the fundamental theorem of groups. (8)
   ii. State and prove the Cayley's theorem for finite group. (8)

   (b) Answer any **TWO**
   i. Define kernel of a homomorphism $f : G \to G'$. Prove that it is a subgroup of G and it is a normal subgroup of G. (6)
   ii. Prove that every subgroup of index 2 of a group G is normal in G. Hence or otherwise prove that A_n is a normal subgroup of S_n. (6)
   iii. If H is a subgroup of group G such that $x^2 \in H$ for every $x \in G$ then prove that H is a normal subgroup of G and G/H is abelian. (6)
   iv. Prove that there are only 2 groups of order 4 upto isomorphism. (6)

2. (a) Answer any **ONE**
   i. Show that characteristic of an integral domain is either 0 or prime. What can be said about the characteristic of field? Justify. (8)
   ii. Let $f : R \to R'$ be ring homomorphism. Show that (8)
      (p) If I is an ideal of R and f is onto then $f(I) = \{f(x) \ : \ x \in I\}$ is an ideal of R',
      (q) If J is an ideal of R', then $f^{-1}(J) = \{x \in R \ : \ f(x) \in J\}$ is an ideal of R.

   (b) Answer any **TWO**
   i. Show that finite integral domain is a field. (6)
   ii. Let R be a finite ring with unity. Show that every non zero element of R is either a zero divisor or a unit. Is the above statement true for infinite commutative ring? Justify. (6)
   iii. Show that the only non-zero ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is identity homomorphism. (6)
   iv. Show that there is no integral domain containing 6 elements. (6)

3. (a) Answer any **ONE**
   i. Define Euclidean domain. (8)
   Show that the ring of Gaussian integers $\mathbb{Z}[i]$, is an Euclidean doamin.
   ii. Define maximal ideal of a ring. Show that an ideal M in a commutative ring R is a maximal ideal if and only if R/M is a field. (8)

[P.T.O.]

(b) Answer any **TWO**

i. Show that a nonzero ideal P of a commutative ring R is prime if and only if $\frac{R}{P}$ is an integral domain (6)

ii. Show that the only maximal ideals in $\mathbb{C}[x]$ are $(x - \alpha)$ for $\alpha \in \mathbb{C}$. (6)

iii. Show that an ideal I in $\mathbb{Z}$ is maximal if and only if $I = p\mathbb{Z}$ for some prime integer p. (6)

iv. Show that ideal $I = \{f(x) \in \mathbb{Z}[x] \,/\, 2|f(0)\}$ is maximal in $\mathbb{Z}[x]$. (6)

4. Answer any **THREE**

(a) If H is the only subgroup of G of the given order then prove that H is a normal subgroup of G. (5)

(b) If a group G is a direct product of two cyclic groups each of order 3 then prove that G is not a cyclic group. (5)

(c) Define zero divisor and unit element in ring R. Show that every element of $\mathbb{Z}_n$ is either a zero divisor or an unit. (5)

(d) Show that if $I_1 \subseteq I_2 \subseteq \cdots$ are ideals of R, then $\cup_{n=1}^{\infty} I_n$ is an ideal of R. (5)

(e) Show that the ring $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{7}]$ are not isomorphic. (5)

(f) Show that 2, 5 are not prime in $\mathbb{Z}[i]$. (5)

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