

3 Hours]

[Total Marks: 100

N.B.: (1) All questions are compulsory.

(2) Figures to the right indicate marks for respective subquestions.

1. Choose the correct option. Attempt all the subquestions.

(i) Let H be a normal subgroup of G . Let $\circ(aH) = 3$ in $\frac{G}{H}$ and $|H| = 10$, (2)
then order of a is

- (a) 1
- (b) 30
- (c) one of 3, 6, 15 or 30
- (d) None of these.

(ii) Which of the following is not true for a normal subgroup H of a group G ? (2)

- (a) $aHa^{-1} \subseteq H$ for each $a \in G$.
- (b) $aHa^{-1} = H$ for each $a \in G$.
- (c) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
- (d) G/H is Abelian.

(iii) Which of the following is not true? (2)

- (a) $\mathbb{Z}_3$ is isomorphic to A_3 .
- (b) $\mathbb{Z}_4$ is isomorphic to $\langle (2\ 1\ 3\ 4) \rangle$, a subgroup of S_4 .
- (c) V_4 is isomorphic to $\{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ a subgroup of S_4 .
- (d) $\mathbb{Z}_6$ is isomorphic to a subgroup of A_4 .

(iv) The group of units of a ring is (2)

- (a) Abelian but may not be cyclic
- (b) Cyclic
- (c) may not be Abelian
- (d) finite

- (v) Consider the ideals of ring of integers $I = 6\mathbb{Z}$ and $J = 10\mathbb{Z}$, then (2)
- $I + J = 22\mathbb{Z}$, $IJ = 120\mathbb{Z}$.
 - $I + J = 2\mathbb{Z}$, $IJ = 60\mathbb{Z}$.
 - $I + J = 2\mathbb{Z}$, $IJ = 30\mathbb{Z}$.
 - None of these.
- (vi) In the polynomial ring $\mathbb{Z}[x]$, consider $I = \{f(x) : f(0) = 0\}$, then (2)
- I is an ideal.
 - I is a maximal ideal.
 - I is ideal but neither prime ideal nor maximal.
 - I is prime ideal but not maximal ideal.
- (vii) Which of the following is true in $\mathbb{Z}[\sqrt{-5}]$ (2)
- $2 + \sqrt{-5}$ is irreducible but not prime.
 - $2 + \sqrt{-5}$ is prime.
 - 3 is prime.
 - $2 + \sqrt{-5}$ is reducible.
- (viii) The number of maximal ideals in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is (2)
- 1.
 - 3.
 - 6.
 - 9.
- (ix) The field of quotients of $\mathbb{Z}[i]$ is (2)
- $\mathbb{Q}[i]$
 - $\mathbb{R}$
 - $\mathbb{C}$
 - None of these.
- (x) Let $I = (x^2 + x + 1)$ in $\mathbb{Z}_n[x]$, $1 \leq n \leq 10$ Then, $\mathbb{Z}_n[x]/I$ is a field if (2)
- $n = 3$
 - for all $n \leq 5$
 - $n = 7$
 - $n = 2, 5$

2. (a) Answer any **ONE**

- (i) Let G and G' be groups and $f : G \rightarrow G'$ be an onto homomorphism. (8)
 Prove that if H is a subgroup G then $f(H) = \{f(h) : h \in H\}$ is a subgroup of G' and $f(Ha) = f(H)f(a)$ for each $a \in G$. Further, if H is normal in G then $f(H)$ is normal in G' . Give example to show that $f(H)$ need not be normal in G' if f is not onto.

- (ii) If $a \in G_1, b \in G_2$ such that $\circ(a) = m, \circ(b) = n$, then prove that (8)
 $(a, b)^k = (a^k, b^k)$ for every $k \in \mathbb{N}$ and $\circ(a, b) = \text{lcm}(m, n)$. Hence
 prove that, G_1, G_2 are cyclic then $G_1 \times G_2$ is cyclic if and only if
 $\circ(G_1)$ and $\circ(G_2)$ are relatively prime.

(b) Answer any **TWO**

- (i) Show that kernel of a group homomorphism $f : G \rightarrow G'$ is a normal (6)
 subgroup of G . Also show that for any normal subgroup H of G
 there is a group homomorphism $\eta : G \rightarrow G/H$ such that $\ker \eta = H$.
 (ii) If $G/Z(G)$ is cyclic then prove that G is Abelian. (6)
 (iii) Show that order of each element of the quotient group $\frac{\mathbb{Q}}{\mathbb{Z}}$ is finite. (6)
 (iv) Show that $\{e, b\}$ is normal in $\{e, b, a^2b, a^2\}$ but not normal in (6)
 $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = e = b^2, aba = b$.

3. (a) Answer any **ONE**

- (i) Define characteristic of a ring R . Show that, characteristic of a (8)
 ring R is n if and only if the order of the multiplicative identity of
 R is n in the group $(R, +)$. Give example of an infinite ring with
 characteristic 2.
 (ii) Let R be a commutative ring. If I, J are ideals in R , Show that (8)
 $I \cap J, I + J$ and IJ are ideals of R , where

$$I + J = \{x + y : x \in I, y \in J\} \text{ and } IJ = \left\{ \sum_{i=1}^n x_i y_i : x_i \in I, y_i \in J, n \in \mathbb{N} \right\}.$$

Further if $R = I + J$, show that $I \cap J = IJ$.

(b) Answer any **TWO**

- (i) Let A be a subring and B be an ideal of a ring R . Then prove that (6)
 $A \cap B$ is an ideal of A and $A/(A \cap B) \simeq (A + B)/B$.
 (ii) Let R, R' be commutative rings and $f : R \rightarrow R'$ be a ring homo- (6)
 morphism. Show that-
 (I) If f is surjective, I is an ideal of R , then $f(I)$ is an ideal of R' .
 (II) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R .
 (iii) Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ and $I = \{a + b\sqrt{2} : a, b \in (6)
 $\mathbb{Z}, a \text{ is even}\}$. Show that the quotient ring R/I is isomorphic to $\mathbb{Z}_2$.
 (iv) Let R be a commutative ring with prime characteristic p and (6)
 $f : R \rightarrow R$ be defined as $f(a) = a^p$ for $a \in R$. Show that f is a ring
 homomorphism.$

4. (a) Answer any **ONE**

- (i) Show that an ideal P in a commutative ring R is a prime ideal if (8)
and only if the quotient ring R/P is integral domain.

Further prove that in a finite commutative ring every prime ideal is maximal.

- (ii) Define irreducible polynomial. (8)

Let F be a field. Show that $F[x]/\langle f(x) \rangle$ is a field if and only if $f(x)$ is irreducible over F .

(b) Answer any **TWO**

- (i) Let R, S be commutative rings. And $f : R \rightarrow S$ be an onto ring (6)
homomorphism. Prove that, if M is a maximal ideal in S then,
 $f^{-1}(M)$ is a maximal ideal in R .
- (ii) Show that the only irreducible polynomials in $\mathbb{R}[x]$ are a linear poly- (6)
nomial $x-a$ or quadratic polynomial x^2+bx+c such that $b^2-4c < 0$,
where $a, b, c \in \mathbb{R}$.
- (iii) Show that in $\mathbb{Z}[i]$, 3 is irreducible but 2 is not irreducible. (6)
- (iv) Show that $\langle x, 2 \rangle$, the ideal generated by x and 2 is a maximal (6)
ideal of $\mathbb{Z}[x]$. Further show that this ideal is not principal ideal.

5. Answer any **FOUR**

- (a) Let G be a group and H be a normal subgroup of G . Then prove that (5)
(p) $(Ha)^n = Ha^n$ for all $n \in \mathbb{Z}$.
(q) $\circ(Ha)$ divides $\circ(a)$.
- (b) Find a subgroup of order 9 in $\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_{15}$. (5)
- (c) Show that a finite field of size 8 has characteristic 2. (5)
- (d) Determine all the ideals of $\mathbb{R}[x]/(x^3 + 3x^2 - 4)$ by stating the results (5)
used.
- (e) Let R be commutative and I, J be ideal of R and P is a prime ideal of (5)
 R that contains $I \cap J$. Prove that either $I \subseteq P$ or $J \subseteq P$.
- (f) Let $\mathbb{F}$ be a field. Show that every ideal of $\mathbb{F}[x]$ is a principal ideal. (5)
