

(3 Hours)

[Total Marks : 100]

- N.B.: 1. All questions are compulsory.  
2. Figures to the right indicate full marks.

Q.1 Choose the correct alternative in each of the following: (20)

- i.  $\lim_{z \rightarrow 0} \frac{\bar{z}}{z} =$   
(a) 1 (b)  $i$  (c)  $-i$  (d) does not exist
- ii. The image of a line under a fractional linear transformation is  
(a) a line (b) a circle (c) a line or a circle (d) None of these
- iii.  $f(z) = \frac{z^2+1}{z^3+9}$  is  
(a) continuous and bounded in  $|z| \leq 2$   
(b) continuous but not bounded in  $|z| \leq 2$   
(c) neither continuous nor bounded in  $|z| \leq 2$   
(d) continuous and bounded everywhere
- iv. If  $u(x, y) = x^2 - y^2, v = 2xy$  then  
(a)  $v$  and  $u$  are harmonic conjugates of each other  
(b)  $u$  is a harmonic conjugate of  $v$  but  $v$  is not a harmonic conjugate of  $u$   
(c)  $v$  is a harmonic conjugate of  $u$  but  $u$  is not a harmonic conjugate of  $v$   
(d) None of these
- v.  $\exp(2 \pm 3\pi i) =$   
(a)  $e^{-2}$  (b)  $-e^2$  (c)  $e^2$  (d)  $e^3$
- vi. If  $e^z = -2$ , then  $z =$   
(a) 0 (b)  $z = \ln 2 + (2n+1)\pi i \quad n = 0, \pm 1, \pm 2, \dots$   
(c)  $i$  (d) none of these
- vii.  $\int_C \frac{e^z}{z-2} dz$ , where  $C$  is the circle  $|z| = 3$ , described in the positive sense is  
(a)  $2\pi i e^2$  (b)  $2\pi i$  (c)  $e^2$  (d) None of these
- viii. Radius of convergence of the series  $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$  is  
(a)  $\infty$  (b) 1 (c) 2 (d) None of these
- ix. The poles of the function  $\frac{\sin z}{\cos z}$  are at  
(a)  $\frac{(2n+1)\pi}{2}, n$  is any integer (b)  $\frac{2n\pi}{3}, n$  is any integer  
(c)  $n\pi, n$  is any integer (d) none of these

- x. The residue of  $f$  at  $z = 0$  where  $f(z) = z \cos \frac{1}{z}$  is  
 (a)  $\frac{1}{2}$  (b)  $-\frac{1}{2}$  (c) 1 (d) none of these

Q.2 a) Attempt any ONE question from the following: (08)

- Let  $f(z) = u(x, y) + iv(x, y)$ . If  $f'(z)$  exists at a point  $z_0 = x_0 + iy_0$ , then prove that the first order partial derivatives of  $u$  and  $v$  exist at  $(x_0, y_0)$  and satisfy Cauchy-Riemann equations  $u_x = v_y$ ,  $u_y = -v_x$ . Also show that  $f'(z) = (u_x)_{z=z_0} + i(v_x)_{z=z_0}$ .
- If  $f'(z_0)$ ,  $g'(f(z_0))$  exist then prove that the function  $F(z) = g(f(z))$  has a derivative at  $z_0$  and  $F'(z_0) = g'(f(z_0))f'(z_0)$ . If  $f: A \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is differentiable at  $z_0 \in A$ , then show that  $f$  is continuous at  $z_0$ . Let  $f: \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$  such that  $f$  is differentiable at  $z_0 \in \Omega$ , then show that  $\exists$  a function  $\eta(z)$  such that  $f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$  where  $\eta(z) \rightarrow 0$  as  $z \rightarrow z_0$ .

b) Attempt any TWO questions from the following: (12)

- Show that  $z(t) = z_0 + tv$  and  $\operatorname{Re}((z - z_0)i\bar{v}) = 0$  represents the same line in  $\mathbb{C}$ .
- If  $f'(z) = 0$  everywhere on a domain  $D$  then show that  $f(z)$  must be constant throughout  $D$ .
- Show that  $f(z) = z|z|$  is differentiable everywhere when  $f$  is treated as a function from  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  but  $\mathbb{C}$  differentiable only at  $z = 0$ .
- If  $f(z) = 8x - x^3 - xy^2 + i(x^2y + y^3 - 8y)$  then determine points at which  $f$  is differentiable,  $f$  is analytic.

Q.3 a) Attempt any ONE question from the following: (08)

- $f$  is analytic inside and on a simple, closed curve  $C$ , taken in the positive sense. Prove that  $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^2}$ . Further state the result generalizing the formula to  $f^n(z)$ .
- Define complex sine and cosine functions. Also establish the following three identities:

$$\begin{aligned} e^z &\neq 0 \quad \forall z \in \mathbb{C} \\ \sin^2 z + \cos^2 z &= 1 \\ |\sinh z|^2 &= \sinh^2 x + \sin^2 y \end{aligned}$$

b) Attempt any TWO questions from the following: (12)

- Evaluate the integral  $\int_C \frac{z+3}{z^2-5z+6} dz$ , where

$$(I) C: |z-2| = \frac{1}{2} \quad (II) C: |z-3| = \frac{3}{4}$$



- iii. Find a Mobius transformation that maps  $i, \infty, 3$  to  $1/2, -1, 3$  respectively.
- iv. State Taylor's theorem and also find Taylor series for  $f(z) = \frac{e^z}{1-z}$  around  $z = 0$ .

Q.4 a) Attempt any ONE question from the following: (08)

- i. If  $C$  is a simple closed curve in the interior of the disc of convergence of the power series  $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  and  $g(z)$  be any function which is continuous on  $C$  then prove that the series  $\sum_{n=0}^{\infty} g(z)a_n(z - z_0)^n$  can be integrated term by term over  $C$  and  $\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} \int_C g(z)a_n(z - z_0)^n dz$ .
- ii. If  $z_1$  is a point inside the circle of convergence  $|z - z_0| = R$  of a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  then show that the series must be uniformly convergent in the closed disk  $|z - z_0| \leq R_1$ , where  $R_1 = |z_1 - z_0|$ .

b) Attempt any TWO questions from the following: (12)

- i. Define the following terms: A removable singularity, A pole of order  $n$ , An essential singularity.
- ii. With the help of a suitable power series, show that  $f(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$  is entire, use it to show that  $\lim_{z \rightarrow 0} \frac{e^z - 1}{z} = 1$ .
- iii. Write Laurent series representations of function  $f(z) = \frac{1}{z(4-z)^2}$  in the domains  $|z| < 4$  and  $|z| > 4$ .
- iv. Evaluate the real improper integral  $\int_0^{\infty} \frac{dx}{x^2+1}$  using the method of residue.

Q.5 Attempt any FOUR questions from the following: (20)

- a) Use Cauchy Riemann equations to check differentiability of  $f(z) = \operatorname{Re} z$
- b) Show that  $u(x, y)$  is harmonic in some domain and find a harmonic conjugate  $v(x, y)$  when  $u(x, y) = x^3 - 3xy^2$
- c) Find image of the set  $\{z \in \mathbb{C} / |z| = \frac{1}{4}, \pi/2 \leq \arg(z) \leq \pi\}$  under the reciprocal map  $w = 1/z$  on the extended complex plane.
- d) Find values of  $z$  such that  $\exp(2z - 1) = 1$ .
- e) Let  $\sum a_n z^n$  has radius of convergence  $R$ . Find the radius of convergence of  $\sum_{n=0}^{\infty} n^3 a_n z^n, \sum_{n=0}^{\infty} a_n z^{3n}$
- f) Evaluate  $\int_C \frac{dz}{(z^2+1)(z^2-4)}$ , where  $C$  is circle  $|z| = 1$ .

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