2 $\frac{1}{2}$ Hours] **[Total Marks: 75**

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. (a) Answer any **ONE**

i. Let G, G' be groups and $f : G \to G'$ be an onto homomorphism. If H' is a subgroup of G' then prove that $f^{-1}(H') = \{h \in H : f(h) \in H'\}$ is a subgroup of G containing ker f. Further show that, if H' is normal in G' then $f^{-1}(H')$ is normal in G. (8)

ii. State and prove the Cayley's theorem for finite groups. (8)

(b) Answer any **TWO**

i. Prove that: $(G_1, \cdot), (G_2, *)$ are cyclic groups and $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ with binary operation $\circ$ defined by $(g_1, g_2) \circ (g_1', g_2') = (g_1 \cdot g_1', g_2 * g_2')$ then $G_1 \times G_2$ is cyclic if and only if $\circ(G_1)$ and $\circ(G_2)$ are relatively prime. (6)

ii. Show the there are two non-isomorphic groups of order 4. (6)

iii. If $G/Z(G)$ is cyclic then prove that G is an Abelian group. (6)

iv. Let $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}, i^2 = j^2 = k^2 = -1 = ijk$. Show that every subgroup of $\mathbb{Q}_8$ is normal in $\mathbb{Q}_8$. (6)

2. (a) Answer any **ONE**

i. State and prove the First Isomorphism Theorem(Fundamental theorem of homomorphism) of rings. (8)

ii. Define characteristic of a ring. Show that the characteristic of an integral domain is either zero or a prime. Give example of a ring with characteristic 0 and a ring with characteristic 5. (8)

(b) Answer any **TWO**

i. Define unit and zero divisor in a ring. Show that every element of $\mathbb{Z}_n$ is either a unit or a zero divisor. (6)

ii. Show that the set of units in a ring R forms a group under multiplication. (6)

iii. Let I be an ideal in a ring R and $\eta : R \to R/I$ be defined by $\eta(a) = a + I$ for $a \in R$. Show that η is a homomorphism and ker $\eta = I$. (6)

iv. Let $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even }\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$. (6)

P.T.O.

3. (a) Answer any **ONE**

   i. Show that the only irreducible polynomials in $\mathbb{R}[x]$ are a linear polynomial $x - a$ or quadratic polynomial $x^2 + bx + c$ such that $b^2 - 4c < 0$, where $a, b, c \in \mathbb{R}$. (8)

   ii. Show that an ideal M in a commutative ring R is a maximal ideal if and only if R/M is a field. (8)

   (b) Answer any **TWO**

   i. Let R be an Integral Domain and $p \in R$. Show that if p is prime then p is irreducible. Is the converse true? Justify your answer. (6)

   ii. For a commutative ring R, prove that R is a field if and only if $\{0\}$ is a maximal ideal in R. (6)

   iii. Prove that the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field, but $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is not a field. (6)

   iv. Show that a field with characteristic p contains a subfield isomorphic to $\mathbb{Z}_p$. (6)

4. Answer any **THREE**

   (a) Show that $\dfrac{\mathbb{R}^*}{\{1, -1\}} \cong \mathbb{R}^+$, for the multiplicative groups $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^+$ of positive reals. (5)

   (b) Let G be a group. Show that the subgroup $H = \{g^2 \,/\, g \in G\}$ of G is normal in G. (5)

   (c) Let R be a ring where $(R, +)$ is cyclic, then show that R is commutative. (5)

   (d) Show that $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are even integers} \right\}$ is an ideal of $M_2(\mathbb{Z})$. (5)
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   (e) Find all ideals of $\mathbb{Z}/12\mathbb{Z}$ using correspondence theorem. (5)

   (f) Show that $x^n - p$ is irreducible in $\mathbb{Q}[x]$ for any prime p. (5)

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