Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. If G is $k-$ critical graph then show that
      I) G is connected
      II) Every vertex v of graph G has atleast $k-1$ degree.
      III) Graph G cannot be partitioned into subgraphs.
   ii. For any simple graph G, prove that $\kappa(G) \leq \kappa^{'}(G) \leq \delta(G)$ where $\kappa(G)$ denote the vertex connectivity and $\kappa^{'}(G)$ denotes the edge connectivity and $\delta(G)$ denotes the minimum degree of a graph G.

   (b) Attempt any **TWO** questions: (12)
   i. Define vertex chromatic number $\chi(G)$. Let G be the graph with p vertices. Show that $\chi(G) \geq \frac{p}{p-\delta(G)}$ where $\chi(G)$ denotes vertex chromatic number of G and $\delta(G)$ denotes minimum degree of G.
   ii. Show that every tree with $n \geq 2$ vertices is 2-chromatic. Is converse true? Justify.
   iii. If G be a connected graph that is not an odd cycle, then prove that G has a 2-edge colouring in which both colours are representing at each vertex of degree at least two.
   iv. Show that if $G_1, G_2, \ldots, G_n$ are n components of graph G then $\pi_k(G) = \prod_{i=1}^{n} \pi_k(G_i)$.

2. (a) Attempt any **ONE** question: (8)
   i. Show that every planar graph is 5 vertex colorable.
   ii. State and prove Max Flow - Min Cut Theorem.

   (b) Attempt any **TWO** questions: (12)
   i. Define dual graph G^* of G. Show that edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.
   ii. Show that there is at least one face of every polyhedron is bounded by an $n-$cycle for some $n = 3, 4$ or 5.
   iii. Show that every planar graph is 6-vertex colorable.
   iv. State and prove Euler theorem for planar graph.

[TURN OVER

3. (a) Attempt any **ONE** question: (8)

   i. Derive the recurrence relation for number of ways of dividing a $n+1-$sided convex polygon into triangular regions by inserting diagonals that do not intersect in the interior and prove using generating function that the solution to this recurrence relation is a Catalan Number.

   ii. Let G be a bipartite graph with bipartition (X, Y). Show that if G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

   (b) Attempt any **TWO** questions: (12)

   i. Define a rook polynomial. Prove that if B is a board of darkened squares that decomposes into the two disjoint sub boards B_1 and B_2 then prove that $R(x, B) = R(x, B_1)R(x, B_2)$, where $R(x, B)$ is a rook polynomial for board B.

   ii. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears, where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

   iii. If $\{A_1, A_2, \ldots, A_n\}$ be a family of set, then prove that the largest number of sets of the family which together have a system of distinct representative equals the minimum value of expression $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + (n-k)$ for all choices of $k = 1, 2, \ldots n$ and all choices of $i_1, i_2, \ldots, i_k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

   iv. How many nonnegative integer solutions are there to the equation
   $x_1 + x_2 + x_3 + x_4 + x_5 = 26$ with $x_i \geq 0$ and $x_i > 0$ for $i = 1, 2, 3, 4, 5$.

4. Attempt any **THREE** questions: (15)

   (a) For any graph G, prove that $\chi(G) \leq \Delta(G) + 1$ where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G. Give an example of graphs for which $\chi(G) < \Delta(G)$.

   (b) If G is a (p, q) graph, then prove that $\chi(G) \geq \frac{p^2}{p^2 - 2q}$ where $\chi(G)$ denotes the vertex chromatic number of G.

   (c) If f is flow in a network N and P is any $f-$incrementing path, then show that there exists a revised flow f' such that $val f' > val f$.

   (d) If G be a simple connected graph with at least 11 vertices then prove that either G or its complement $\overline{G}$ must be nonplanar.

   (e) Solve recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ for all $n \geq 2$ subject to initial conditions $a_0 = 1$ and $a_1 = 1$ using generating function.

   (f) Let $A = (A_1, A_2, A_3, A_4, A_5, A_6)$, where $A_1 = \{1, 2, 3\}, A_2 = \{1, 2, 3, 4, 5\}, A_3 = \{1, 2\}, A_4 = \{2, 3\}, A_5 = \{1\}, A_6 = \{1, 3, 5\}$. Does family A have an System of Distinct Representative? If not, what is the largest number of sets in the family with an System of Distinct Representative?

****