Duration:[3 Hours] **[Total Marks: 100]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. Let c be a proper coloring of G = (V, E) using t colors, then the coloring partitions V into
(a) $t-1$ parts (b) t parts
(c) one part (d) 2 parts

ii. If graph G is k-critical then
(a) G is acyclic (b) G is disconnected
(c) G is connected (d) all of these.

iii. Which of the following can be a chromatic polynomial?
(a) $k^4 - 3k^3 + 3k^2 - k$ (b) $3k^3 - 4k^2 + k$
(c) $k^4 - 5k^3 + 7k^2 - 6k + 3$ (d) $k^3 + k^2 + k$

iv. K_n is planar if
(a) $n > 4$ (b) $n \leq 4$ (c) $n = 5$ (d) None of these

v. A connected planar graph has an equal number of vertices and faces. If there are 20 edges in this graph, the number of vertices must be:
(a) 9 (b) 10
(c) 20 (d) 11

vi. If f is a flow in a network N and P be any $f-$incrementing path with tolerance $\epsilon(P) > 0$, then define a new flow f' as follows : $f'(a) = f(a) + \epsilon(P)$ for an forward arc $a \in P$, $f'(a) = f(a)$ for an backward arc $a \in P$ and $f(a) = f(a)$ for other arcs a of N.Then value of f' equals to
(a) $valf + \epsilon(P)$ (b) $valf - \epsilon(P)$
(c) same as $valf$ (d) $valf$x$\epsilon(P)$

vii. Let $R(x, B)$ denotes the rook polynomial for the board B of darkened squares consisting of m rows and n columns, then
(a) constant term is 1
(b) coefficient of x^k is number of ways of placing k non capturing rooks
(c) $r_k(B) = 0$ if $k > min\{m, n\}$
(d) all of the above.

viii. The number of ways to climb a staircase with 12 steps taking 1 or 2 steps at a time is
(a) 987 (b) 610
(c) 377 (d) 233

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ix. The number of different system of distinct representatives for the family $A_1 = \{2,3,4\}$, $A_2 = \{1,3,4\}$, $A_3 = \{1,2,4\}$, $A_4 = \{1,2,3\}$ is

(a) 4! (b) 3! (c) 9 (d) None of these

x. If M is maximum matching then which one of the following statement is true?

(a) There does not exists any matching M' such that $|M'| < |M|$
(b) There does not exists any matching M' such that $|M'| > |M|$
(c) There exists any matching M' such that $|M'| > |M|$
(d) None of these

2. (a) Attempt any **ONE** question from the following: (8)

i. For a simple graph G of order p and size q, prove that $\pi_k(G)$, the chromatic polynomial of the graph G, is monic polynomial of degree p in k with integer coefficients and constant term zero. Further prove that its coefficients are alternate in sign and the coefficient of k^{p-1} is $-q$.

ii. For any simple graph G, prove that $\kappa(G) \leq \kappa^{'}(G) \leq \delta(G)$ where $\kappa(G)$ denote the vertex connectivity and $\kappa^{'}(G)$ denotes the edge connectivity and $\delta(G)$ denotes the minimum degree of a graph G.

(b) Attempt any **TWO** questions from the following: (12)

i. State Vizing theorem for edge coloring of graphs. Show that $\chi'(G) \geq \Delta(G)$ where $\chi'(G)$ denotes edge chromatic number and $\Delta(G)$ denotes the maximum degree of G. Give an example of the graph for which $\chi'(G) = \Delta(G)$.

ii. Let $\pi_k(G)$ denote the chromatic polynomial of the graph G. If G is simple graph then prove that $\pi_k(G) = \pi_k(G - e) - \pi_k(G.e)$ where e is an edge of G.

iii. If G is a graph with p vertices and $\overline{G}$ is its complement of G, then show that $\chi(G) + \chi(\overline{G}) \leq p + 1$, where $\chi(G)$ is the vertex chromatic number of graph G.

iv. If G is a (p,q) graph, then prove that $\chi(G) \geq \frac{p^2}{p^2-2q}$ where $\chi(G)$ denotes the vertex chromatic number of G.

3. (a) Attempt any **ONE** question from the following: (8)

i. Show that every planar graph is 5 vertex colorable.

ii. State and prove Max Flow - Min Cut Theorem.

(b) Attempt any **TWO** questions from the following: (12)

i. Show that there is at least one face of every polyhedron is bounded by an $n-$cycle for some $n = 3, 4$ or 5.

ii. Show that the edge e is a loop in G if and only if e^* is a bridge in G^* where G^* is dual of graph G.

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iii. Define a value of flow and capacity of cut in network N. If f is any flow and K be any cut in a network N then show that $val(f) \leq cap(K)$.

iv. If G is a connected simple planar graph with $p \geq 3$ vertices, q edges and f regions then
I) Show that if $q = 3p - 6$ then each region is triangle.
II) Deduce that a convex polyhedron with 12 vertices and 20 faces is composed entirely of triangles.

4. (a) Attempt any **ONE** question from the following: (8)

i. State and prove the necessary and sufficient condition for a family of n sets to have System of Distinct Representative.

ii. An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for a_n, the number of different ways for the elf to ascend the $n-$stair staircase and solve it by using generating function.

(b) Attempt any **TWO** questions from the following: (12)

i. Define a rook polynomial. Let $R_{n,m}(x)$ be the rook polynomial for the $n \times m$ chess board, all squares may have rooks. Show that $R_{n,m}(x) = R_{n-1,m}(x) + mxR_{n-1,m-1}(x)$

ii. Show that a matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.

iii. Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \dots)^5$. What is the coefficient of x^r?

iv. Let h_n denote the number of nonnegative integral solutions of the equation $3e_1 + 4e_2 + 2e_3 + 5e_4 = n$. Find the generating function $f(x)$ for $h_0, h_1, \dots, h_n, \dots$

5. Attempt any **FOUR** questions from the following: (20)

(a) For any graph G, prove that $\chi(G) \leq \Delta(G) + 1$ where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G. Give an example of graphs for which $\chi(G) < \Delta(G)$.

(b) Prove that every tree with $n \geq 2$ vertices is 2-chromatic.

(c) If G be a simple connected graph with at least 11 vertices then prove that either G or its complement $\overline{G}$ must be nonplanar.

(d) If f is flow in a network N and P is any $f-$incrementing path, then show that there exists a revised flow f' such that $val f' > val f$.

(e) Find the rook polynomial for the following

$$\{(1,1), (2,5), (3,3), (4,2), (4,4), (5,1), (5,3)\}.$$

(f) Let $\{A_1, A_2, \dots, A_n\}$ be a family of sets such that for each k, $1 \leq k \leq n$ and for each choice of $1 \leq i_1 < i_2 < \dots < i_k \leq n$, $|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}| \geq k + 1$. Let x be any element of A_1. Show that $\{A_1, A_2, \dots, A_n\}$ has a system of distinct representatives in which x represents A_1.

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