REVISED COURSE [Max Marks:75]

Duration: $2^1/_2$ Hours

N.B. 1. All questions are compulsory.

2.From Question 1,2 and 3, Attempt any one from part(a) and any two from part(b).

3. From Question 4, Attempt any THREE

4. Figures to the right indicate marks for the respective parts.

Q.1 a i Let $< f_n >$ be sequence of differentiable real valued functions on [a, b] such that $< f_n(x_0) >$ converges for some $x_0 \in (a, b)$ and $< f'_n >$ converges uniformly to function g on $[a, b]$. Prove that $< f_n >$ converges uniformly on $[a, b]$ and if f is uniform limit of $< f_n >$ then f is differentiable on (a, b) and $f' = g$ on (a, b). (8)

ii State and prove Weierstrass M- test.

b i State and prove Cauchy's criterion for uniform convergence of the sequence $< f_n >$ of functions of real numbers. (12)

ii Examine whether $\int_0^1 \sum_0^\infty x^n(1-2x^n)dx = \sum_0^\infty \int_0^1 x^n(1-2x^n)dx$. Is the series $\sum_0^\infty x^n(1-2x^n)$ uniformly convergent in [0, 1]? Justify.

iii Find M_n, where $M_n = Sup\left\{\frac{x}{(n+x^2)^2} : x \in [a, b]\right\}$, using Weierstrass M- test. Evaluate $\int_a^b \sum_0^\infty \frac{x}{(n+x^2)^2} dx$.

iv Let $f_n: [0, 1] \to IR$ be given by $f_n(x) = x^n$. Let f be pointwise limit of $< f_n >$. Is f continuous on $[0, 1]$. Does $< f_n >$ converge uniformly on $[0, 1]$? Justify.

Q.2 a i If a function f is continuous throughout a region R that is closed and bounded then show that there exists a non-negative integer M such that $|f(z)| \le M \ \forall z \in R$. Also show that if $f'(z_0), g'(z_0)$ exist , $g'(z_0) \neq 0$, $f(z_0) = 0 = g(z_0)$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$. (8)

ii Let $f(z) = u(x, y) + iv(x, y)$. If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$ then prove that the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy Cauchy- Riemann equations $u_x = v_y$, $u_y = -v_x$. Show that the converse is not true. Also show that $f'(z) = (u_x)_{z=z_0} + i(v_x)_{z=z_0}$.

b i Using the definition, discuss differentiability of the function $f(z) = z^2$ at any $z \in \mathbb{C}$. (12)

ii f is analytic throughout on a given domain D. If $|f(z)|$ is constant on D, show that $f(z)$ must be constant on D.

iii If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then show that its component functions u and v are harmonic in D.

iv Find the image of the given set under the reciprocal map $w = \frac{1}{z}$ in the extended complex plane : $\frac{1}{5} \le |z| \le 2$

(P.T.O)

Q.3 a i State and prove Cauchy Integral Theorem. (8)

ii Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $|z - z_0| < R_0$ where $a_n = \frac{f^n(z_0)}{n!}$.

b i If a function f is analytic at a given point then show that its derivatives of all orders are analytic at that point too. Further suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R and if M_R denotes the maximum value of $|f(z)|$ on C_R then show that $|f^n(z_0)| \leq \frac{n! M_R}{R^n}, n = 1,2,3, \dots$ (12)

ii Prove that a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = r$.

iii Show that any singular point of the function $f(z) = \frac{e^z}{z^2 + \pi^2}$ is a pole. Further determine the order m of each pole and find the corresponding residue of f.

iv State Laurent's Theorem. For $f(z) = \frac{-1}{(z-1)(z-2)}$, write Laurent series expansion in the domains : $|z| < 1$, $2 < |z| < \infty$.

Q.4 i Does the sequence $< f_n >$,where $f_n(x) = \frac{nx}{1+nx^2}$ converges uniformly on $[0, \infty)$? Justify. (15)

ii For $|x| < 1$, show that $tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^n}{2.4.6 \dots 2n} \frac{x^{2n+1}}{2n+1}$.

iii Test differentiability of the function $f(z) = z \ Im\, z$ at $(0, 0)$.

iv Construct a linear fractional transformation that maps the points $i, \infty, 3$ to $\frac{1}{2}, -1, 3$ respectively.

v Evaluate $\int_C \frac{1}{(z-z_0)^{n+1}} dz$ where C is the circle $|z - z_0| = r$, n is a non zero integer using a parameterisation of C.

vi Evaluate $\int_C \frac{sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz$ where $C : |z| = 2$.