

- N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following:

- (i) In a metric space (X, d) , prove that arbitrary union of open sets is open in X . Give an example to show that arbitrary intersection of open sets is not open in X . (8)
(ii) Let (X, d) be a metric space and $A, B \subseteq X$. Show that
(I) $A \subseteq B \implies A^\circ \subseteq B^\circ$
(II) $(A \cap B)^\circ = A^\circ \cap B^\circ$
(III) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

(b) Attempt any Two of the following:

- (i) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X . If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2) .
(ii) (\mathbb{Z}, d) and (\mathbb{Z}, d_1) are metric spaces where d is the usual metric (induced from \mathbb{R}) and d_1 is the discrete metric in \mathbb{Z} . Prove that d and d_1 are equivalent metrics.
(iii) Let d_1, d_2 be metrics on X . Define $d : X \times X \rightarrow \mathbb{R}$ as $d(x, y) = \max \{d_1(x, y), d_2(x, y)\}$. Show that d is a metric on X .

- (iv) $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms on \mathbb{R}^2 where for $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\|_1 = \sum_{i=1}^2 |x_i|$, $\|x\|_2 =$

$$\sqrt{\sum_{i=1}^2 x_i^2}. \text{ Show that } \|x\|_2 \leq \|x\|_1 \text{ and } \|x\|_1 \leq \sqrt{2} \|x\|_2 \text{ for } x \in \mathbb{R}^2$$

2. (a) Attempt any One of the following:

- (i) Show that for a subset F of a metric space (X, d) , the following statements are equivalent:
(I) F is closed
(II) F contains all its limit points.
(ii) Let (X, d) be a metric space and A be a subset of X . Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p .

(b) Attempt any Two of the following:

- (i) Let $A, B \subset \mathbb{R}$ (distance being usual), where $A = \mathbb{N}$ and $B = \left\{ n + \frac{1}{n} : n \in \mathbb{N}, n > 1 \right\}$.

Find $d(A, B)$

- (ii) Let A be a subset of a metric space (X, d) . Prove that

- (I) $\overline{(X \setminus A)} = X \setminus A^\circ$
(II) $(X \setminus A)^\circ = X \setminus \overline{A}$

- (iii) Show that $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a closed subset of \mathbb{R}^2 , where the distance being Euclidean.

- (iv) Prove that a subset A of a metric space (X, d) is dense in X if and only if $G \cap A \neq \emptyset$ for each non-empty open subset G of X .

(8)

3. (a) Attempt any One of the following:

- If $K \subseteq \mathbb{R}^n$ is such that K is compact then prove that K has Bolzano-Weierstrass Property. (distance being Euclidean).
- Show that a compact subset of (\mathbb{R}^n, d) where d Euclidean, is closed and bounded. Give an example to show that a closed and bounded subset of a metric space is not compact.

(12)

(b) Attempt any two:

- Prove that a subset of a discrete metric space is compact if and only if it is finite.
- (X, d) is a metric space and (x_n) is a sequence in X such that (x_n) converges to some point $p \in X$. If $S = \{x_n : n \in \mathbb{N}\} \cup \{p\}$ then show that S is compact by using the definition of compactness.
- Let A, B be compact subsets of (\mathbb{R}, d) , distance d being usual. Show that $A \times B$ is a compact subset of (\mathbb{R}^2, d') where d' is the Euclidean distance.
- Consider the metric space (\mathbb{R}, d) , where d is the usual distance. Show that $\{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. Is $(0, 1)$ compact? Justify your answer.

(15)

4. Attempt any Three of the following:

- Prove or disprove : If (X, d) be a metric space and $x, y \in X, r, s > 0$ and $B(x, r) = B(y, s)$, then either $x = y$ or $r = s$
- Show that $\| \cdot \|$ is a norm on X , where $X = M_2(\mathbb{R})$ and $\|A\| = \max \{|a_{ij}| : 1 \leq i, j \leq 2\}$ for $A = (a_{ij}) \in X$
- Let (X, d) be a discrete metric space and $A \subseteq X$. Then prove that $\overline{(A^\circ)} = \overline{A}$
- Consider the sequence (f_n) of functions in $C[0, 1]$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \text{if } \frac{1}{2} - \frac{1}{n} < t \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

Show that $\{f_n\}$ is Cauchy w.r.t. $\| \cdot \|_1$ where $\|f\|_1 = \int_0^1 |f(t)| dt$

- Let $A = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$. Determine whether A is compact. Justify your answer.
- Prove or disprove : A closed ball $B[x, r]$ in a metric space is compact.