TYBSC Sem-VI 13/4/17
Metric Topology
2016-2017

Q. P. Code: 05027

(7)

3. (a) Attempt any One from the following: (8)

(i) (X, d) is a metric space and A, B are subsets of X such that A is connected and $A \subseteq B \subseteq \overline{A}$. Prove that B is connected. Give an example to show that if $A \subseteq B \subseteq C \subseteq X$ and A, C are connected then B need not be connected.

(ii) Prove that a path connected subset of $\mathbb{R}^n$ (distance being Euclidean) is connected.

(b) Attempt any Two from the following: (12)

(i) If (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ (distance in $\mathbb{Z}$ being usual distance) is a continuous function then prove that f is a constant function.

(ii) Show that the set $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, 1 < y < 5\}$ is a convex set in $(\mathbb{R}^2, d)$ where d is the Euclidean distance.

(iii) (X, d) is a connected metric space which is not bounded. Prove that for each $x_0 \in X$ and for each $r > 0$ the set $\{x \in X : d(x, x_0) = r\}$ is nonempty.

(iv) Let A be the union of the following subsets S and L of $\mathbb{R}^2$. Show that A is connected. (distance in $\mathbb{R}^2$ being Euclidean).

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$
$$L = \{(x, y) \in \mathbb{R}^2 : x \geq 1 \text{ and } y = 0\}$$

4. Attempt any Three from the following: (15)

(a) $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous such that f takes only rational values then show that f is a constant function.

(b) Prove that $(0, 1)$ as a subspace of $(\mathbb{R}, d)$ (d being usual distance) is not complete but is complete as a subspace of $(\mathbb{R}, d_1)$ where d_1 is discrete metric.

(c) Let $f : [a, b] \rightarrow [a, b]$ is continuous on $[a, b]$ and differentiable on (a, b). If $\exists c \in \mathbb{R}$ with $0 < c < 1$ such that $|f'(x)| \leq c$, $\forall x \in (a, b)$ then prove that f is a contraction of $[a, b]$.

(d) Let (X, d) and (Y, d') be metric spaces. Show that if $f : X \longrightarrow Y$ is uniformly continuous on X and if (x_n) in X is Cauchy then show that the sequence $(f(x_n))$ is Cauchy in Y.

(e) Show that $E = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 - y^2 = 1\}$ is path connected.

(f) Prove or disprove : If A, B are connected subsets of $\mathbb{R}$ with respect to usual distance and $A \cap B \neq \emptyset$, then $A \cap B$ is also connected.

****

(6)

Q. P. Code: 05027

Duration 2 ½Hrs REVISED COURSE Marks: 7

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following:
   (i) If in a metric space (X, d), for every decreasing sequence $\{F_n\}$ of non-empty close sets with $d(F_n) \longrightarrow 0$, we have $\cap_{n\in\mathbb{N}} F_n$ is a singleton set then prove that (X, d) complete.
   (ii) If $x, y \in \mathbb{R}$ are such that $x < y$ then show that there exists a rational number $r \in$ (such that $x < r < y$.

   (b) Attempt any Two from the following:
   (i) Prove that the set of real numbers $\mathbb{R}$ is complete with respect to the usual distance
   (ii) Define a complete metric space. If (X, d) is a complete metric space and Y is a close subspace of X then prove that (Y, d_Y) is complete.
   (iii) Check if Cantor's Theorem is applicable in the following examples. Also , find $\cap_{n\in\mathbb{N}} F$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usua
   (I) $F_n = [n, \infty)$
   (II) $F_n = (0, \frac{1}{n})$
   (iv) Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = (x-a)^2(x-b)^2 + x$ takes th value $\frac{a+b}{2}$ for some value of $x \in \mathbb{R}$. (distance in $\mathbb{R}$ being usual.)

2. (a) Attempt any One from the following:
   (i) Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Prove that f is continuous on X if and only for each open subset G of Y, $f^{-1}(G)$ is an open subset of X.
   (ii) Let (X, d) be a complete metric space. If $T : X \longrightarrow X$ is a contraction, then pro that T has a unique fixed point, that is , $\exists$ a unique point $x \in X$ such that $T(x) =$

   (b) Attempt any Two from the following:
   (i) (X, d) and (Y, d') be metric spaces. If $f : (X, d) \longrightarrow (Y, d')$ is continuous on X the prove that for every $A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$. Also show that the inequality may l strict.
   (ii) Prove that every function $f : (\mathbb{N}, d) \longrightarrow (Y, d')$ where d is the usual metric and $(Y, d$ is any metric space is continuous.
   (iii) Discuss the uniform continuity of $f : [1, \infty) \longrightarrow \mathbb{R}$, defined by $f(x) = \frac{1}{x}$.
   (iv) Let $f : X \longrightarrow (0, \infty)$ be a continuous function, where (X, d) is a compact metr space. Show that $\exists\ \epsilon > 0$ such that $f(x) \geq \epsilon,\ \forall\ x \in X$.

(P.T.O)