Duration 2 $\frac{1}{2}$Hrs OLD COURSE Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One question: (8)
   (i) Let f be a continuous real valued periodic function, defined on $[-\pi, \pi]$ and having period 2π. If $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series of f on $[-\pi, \pi]$ then prove that : $S_n(x) - f(x) = \frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right]D_n(t)\mathrm{d}t$ where $D_n(x)$ is the Dirichlet's kernel.
   (ii) State and prove Bessels Inequality.

   (b) Attempt any Two questions: (12)
   (i) For $f(x) = x\cos x, x \in [-\pi, \pi]$ and $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. find Fourier coefficients a_0, a_1, b_1.
   (ii) Let $f \in C[-\pi, \pi]$ and f has Fourier sereis $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, show that $\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)(a_k \cos kt + b_k \sin kt)$
   (iii) Define Fejer's Kernel $K_n(t)$. Prove that $K_n(t) = \frac{\sin^2(\frac{nt}{2})}{2n\sin^2\frac{t}{2}}$ $-\infty < t < \infty$, $t \neq 2k\pi, k \in \mathbb{Z}, t \in \mathbb{R}$
   (iv) Is the series $\sum_{n=1}^{\infty}\left[\frac{\cos nx + \sin nx}{n^{\frac{3}{2}}}\right]$ the Fourier series of a function $f \in C[-\pi, \pi]$? Justify your answer.

2. (a) Attempt any One question: (8)
   (i) $K \subseteq \mathbb{R}^n$ (distance Euclidean), K is closed and bounded. Show that K is sequentially compact.
   (ii) Let $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ (distance Euclidean). Prove that I is compact.

   (b) Attempt any Two questions: (12)
   (i) Let (X, d) and (Y, d') be metric spaces. If (X, d) is compact and $f : X \longrightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y.
   (ii) Show that $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is a compact subset of $(\mathbb{R}, d)$ where d is usual distance in $\mathbb{R}$ using the definition of a compact subset.
   (iii) Let (X, d) be a compact metric space and $f : X \longrightarrow (0, \infty)$ is continuous. Show that there exists $\epsilon > 0$ such that $f(x) \geq \epsilon\ \forall x \in X$.

(P.T.O)

(iv) Prove that a subset of a discrete metric space is compact if and only if it is finite.

3. (a) Attempt any One question: (8)

(i) Show that a subset $E \subseteq \mathbb{R}$ is connected if and only if E is an interval (distance being usual).

(ii) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \longrightarrow \{1, -1\}$ is constant.

(b) Attempt any Two questions: (12)

(i) Let (X, d) be a metric space and A be a connected subset of X. If $A \subseteq B \subseteq \bar{A}$, then show that B is connected. In particular, prove that $\bar{A}$ is connected.

(ii) Show that $S = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ is not connected. Hence show that S not path connected in a Euclidean metric space $\mathbb{R}^2$.

(iii) If (X, d) is a connected metric space and $f : X \longrightarrow \mathbb{Z}$ a continuous function, prove that f is constant. (distance in Z being usual).

(iv) Show that a convex set in $\mathbb{R}^n$ is path connected (distance being Euclidean).

4. Attempt any Three from the following: (15)

(a) If the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges uniformly to f on $[-\pi, \pi]$ then prove that Fourier series of f is $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

(b) $f(x) = \frac{x^2}{4}$, $-\pi \leq x \leq \pi$. Find the Fourier series of f. Assuming that the Fourier series of f converges to $f(x)$ at $x = 0$, find the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(c) Show that a closed subset of a compact set is compact in any metric space.

(d) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of $\mathbb{R}^2$, distance being Euclidean.

(e) Let A and B be path connected subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path connected.

(f) Let (X, d) be a connected metric space which is not bounded. Prove that for each $x_0 \in X$ and each $r > 0$, the set $\{x \in X : d(x, x_0) = r\}$ is non-empty.