Analysis & Multivariable Calculus-II

Q. P. Code: 04987

ation: $2\frac{1}{2}$ Hours OLD COURSE **Max. Marks : 75**

1) All questions are compulsory
2) Figures to the right indicate marks.

1 (a) Attempt any ONE of the following **(8)**

(i) If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and if $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ both exists, then prove that
$\lim_{x\to a}(\lim_{y\to b} f(x,y)) = \lim_{y\to b}(\lim_{x\to a} f(x,y)) = L.$
Give an example to show that the converse is not true.

(ii) Let S be an open subset of $\mathbb{R}^n$ and f, g : S → $\mathbb{R}^m$ and let a ∈ S, $\lambda \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ then using $\epsilon - \delta$ definition prove that p) $\lim_{x\to a}(\lambda f(x)) = \lambda b$ q) $\lim_{x\to a}(f(x).g(x)) = b.c$

(b) Attempt any TWO of the following **(12)**

(i) Let f: $\mathbb{R}^2 \to$: $\mathbb{R}^2$ be defined by $f(x, y) = (x^2 - y^2, x^2 + y^2)$. Using $\epsilon - \delta$ definition show that each component of f is continuous at (1, 2).

(ii) If f: $\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = |x| + |y|$ then show that $f_x(0,0)$ *and* $f_y(0,0)$ do not exist. Check whether f is continuous at (0, 0)

(iii) Find $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sqrt{(x^2+y^2)}}$ using polar co-ordinates.

(iv) Using $\varepsilon - \delta$ definition, discuss the continuity of $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^2+2y, y^2+2x)$ at (2, 3)

Q.2 (a) Attempt any ONE of the following **(8)**

(i) Let S be an open subset of $\mathbb{R}^2$ and $f: S \to \mathbb{R}$ be such that $D_1f, D_2f, D_{12}f, D_{21}f$ exists on S. If $(a,b) \in S$ and $D_{12}f, D_{21}f$ are continuous on S, then show that $D_{12}f(a,b) = D_{21}f(a,b)$.

(ii) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}$ be differentiable at $a \in S$ with total derivative $Df(a)$. Show that $f'(a;y)$ exists for all $y \in \mathbb{R}^n$ and $f'(a;y) = Df(a)$ and $f'(a;y) = \sum_{k=1}^{n} D_kf(a)y_k = \nabla f(a) \cdot y$ for all $y \in \mathbb{R}^n$.

(b) Attempt any TWO of the following **(12)**

(i) State and prove the Mean Value Theorem for a scalar field

(ii) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Let A(1,3), B(3,3), C(1,7), D(6,15). The directional derivative of f at A in the direction of AB is 3 and in the direction of AC is 26. Find the directional derivative of f at A in the direction of AD.

(iii) If $f: \Re^2 \to \Re^2, g: \Re^2 \to \Re^3$ $f(x,y) = (xy^2, xy)$, $g(u,v) = (u+v, u-v, uv)$.
(p)Compute the Jacobian matrices $Df(x,y), Dg(u,v)$ & $D(g\circ f)(x,y)$
(q)Verify that $D(g\circ f)(1,1) = Dg(1,1)\cdot Df(1,1)$

(iv) Let $z = e^{u+v+w}$, where $u = x^2\sin^2 y$, $v = 2x\sin x\sin y$, $w = y^2$. Use chain rule to find z_x, z_y.

Q.3 (a) Attempt any ONE of the following **(8)**

(i) State and prove Stoke's Theorem for an oriented smooth simple parametrized surface in $\Re^3$ bounded by simple ,closed, curve traversed counter clockwise assuming general form of Green's Theorem.

(ii) State Divergence Theorem for a solid in 3 – space bounded by an orientable closed surface with positive orientation and prove the Divergence theorem for cubical region.

(P.T.O)

(12)

(b) Attempt any TWO of the following

(i) Compute the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$

(ii) Evaluate the surface integral $\iint_S \bar{F}\cdot\hat{n}dS$ if $\bar{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the sphere $x^2 + y^2 + z^2 = 1$.

(iii) Use Stokes' Theorem to compute the integral $\iint_s curl\,\bar{F}\cdot\hat{n}ds$, where $\bar{F}(x,y,z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2=1$ above the xy –plane.

(iv) Verify Divergence Theorem for vector field $\bar{F}(x,y,z) = 3x\hat{i} + xy\hat{j} + 2xz\hat{k}$ and V is the cube bounded by the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.

Q.4 Attempt any THREE of the following **(15)**

(i) If $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, then find $\lim_{x\to 0}(\lim_{y\to 0} f(x,y))$ and $\lim_{y\to 0}(\lim_{x\to 0} f(x,y))$. Also find $\lim_{(x,y)\to(0,0)} f(x,y)$ if exists

(ii) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right) & ,if\ xy \neq 0 \\ 0 & ,if\ xy = 0 \end{cases}$$

Find $D_{12}f(0,0), D_{21}f(0,0)$ and check whether they are equal.

(iii) Find all differentiable vector fields $f: \Re^3 \to \Re^3$ for which the Jacobian matrix $Df(x,y,z)$ = diag (p(x) , q(y) , r(z)) where p, q, r : $\Re \to \Re$ are continuous functions .

(iv) Given $u = f(x,y)$ has continuous partial derivatives with respect to x and y. If $x = r\,cos\theta, y = r\,sin\theta$, then show that $u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2}\, u_\theta^2$.

(v) Evaluate the surface integral $\iint_S xzdS$,where S is the triangle with the vertices (1,0,0), (0,1,0) and (0,0,1).

(vi) Define the Fundamental Vector Product for a surface S whose vector equation is $\bar{r}(u,v) = X(u,v)\hat{\imath} + Y(u,v)\hat{\jmath} + Z(u,v)\hat{k}$ $\forall$(u,v) $\in$ T in uv-plane. Compute $\left\|\frac{\partial\bar{r}}{\partial u} x \frac{\partial\bar{r}}{\partial v}\right\|$ for $\bar{r}(u,v) = vsin\alpha cosu\hat{\imath} + vsin\alpha sinu\hat{\jmath} + vcos\alpha\hat{k}$