TY BSC Sem-VI 12/6/17 Algebra 65

Algebra

2016-2017

Q. P. Code: 05022

extra

2 $\frac{1}{2}$ Hours] Revised course [Total Marks: 75

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N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. (a) Answer any **ONE**
   i. Let H be a subgroup of group G. Prove that the following statements are equivalent. (8)
      (p) $aHa^{-1} \subseteq H$ for each $a \in G$.
      (q) $aHa^{-1} = H$ for each $a \in G$.
      (r) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
      (s) $HaHb = Hab$ for each $a, b \in G$.
   ii. State and prove the Cayley's theorem for finite group. (8)

   (b) Answer any **TWO**
   i. Let H, K be normal subgroups of G and H be a subgroup of K. Prove that $\frac{G/H}{K/H} \simeq \frac{G}{K}$. (6)
   ii. If a cyclic group H of a group G is normal in G, then show that every subgroup of H is normal in G. (6)
   iii. Find the order of each element of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Is $\mathbb{Z}_2 \times \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_8$? Justify. (6)
   iv. Suppose G is a non-abelian group of order p^3 where p is a prime and $Z(G) \neq \{e\}$, then prove that $|Z(G)| = p$. (6)

2. (a) Answer any **ONE**
   i. Let R, R' be commutative rings and $f : R \to R'$ be a ring homomorphism. Show that- (8)
      (p) If f is surjective, I is an ideal of R, then $f(I)$ is an ideal of R'.
      (q) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R.
   ii. Show that, characteristic of a ring R is n if and only if the order of the multiplicative identity of R is n in the group $(R, +)$. Further if char $R = n$, where R is an integral domain, then show that n is a prime. (8)

   (b) Answer any **TWO**
   i. Let A be a subring and B be an ideal of a ring R. Then prove that $A \cap B$ is an ideal of A and $A/(A \cap B) \simeq (A + B)/B$. (6)
   ii. Let R be a commutative ring. Show that $I = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal of R. Also show that R/I has no nilpotent element. (6)
   iii. Show that there is exactly one non-zero ring homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$. (6)
   iv. Show that, if R is a ring having 6 elements then R is commutative. Is R an integral domain? Justify. (6)

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3. (a) Answer any **ONE**

   i. Let F be a field and $p(x) \in F[x]$. Show that $< p(x) >$ is maximal ideal of $F[x]$ if and only if $p(x)$ is irreducible.

   ii. Let R be an Integral Domain , Let $p \in R$. Then ,

   (p) p is prime iff (p) is a non zero prime ideal of R.

   (q) If p is prime then p is irreducible.

   (b) Answer any **TWO**

   i. State and prove the Eisenstein's criteria for irreducibility of a polynomial $f(x)$ over $\mathbb{Q}$.

   ii. Let F be a field. Show that every ideal of $F[x]$ is a principal ideal.

   iii. Let R be a commutative ring and I, J be any two ideals of R. Let P be a prime ideal of R such that $IJ \subseteq P$. Prove that $I \subseteq P$ or $J \subseteq P$.

   iv. Show that ideal $I = \{a + bi \ : \ a, b \in \mathbb{Z}, a \bmod 2 = b \bmod 2\}$ is a maximal ideal of $\mathbb{Z}[i]$.

4. Answer any **THREE**

   (a) Let H be a normal subgroup of a finite group G . If G/H has an element of order n then show that G has an element of order n.

   (b) Let $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}_3 \right\}$. Show that G is a subgroup of $GL_3(\mathbb{Z}_3)$. Also Show that G is abelian of order 9 and is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

   (c) Let R be a commutative ring. If u is a unit and a is nilpotent in R, show that $u + a$ is a unit in R.

   (d) Show that every element of a finite commutative ring is either a unit or a zero divisor.

   (e) Let R, S be commutative rings and $f : R \to S$ be an onto ring homomorphism. Prove that if M is a maximal ideal in S, $f^{-1}(M)$ is a maximal ideal in R.

   (f) Prove that the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field, but $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is not a field.

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