TY BSc Sem-VI 2016-2017
Algebra-II (60:40) (75-25) 9

Q. P. Code: 04990

2 $\frac{1}{2}$ Hours] **Old Syllabus** [Total Marks: 75

N.B.: (1) **All questions are compulsory.**
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. State and prove the first isomorphism theorem for groups. (8)
   ii. H be a subgroup of G, then prove that $a^{-1}Ha \subseteq H$ for all $a \in G$ if and only if $HaHb = Hab$ for all $a, b \in G$. (8)

   (b) Answer any **TWO**
   i. If H_1, H_2 are normal subgroups of G_1, G_2 respectively then prove that $H_1 \times H_2$ is normal in $G_1 \times G_2$. (6)
   ii. Show that any two cyclic groups of same order are isomorphic. (6)
   iii. Let $f : G \to G'$ be a group homomorphism. Prove that if H is normal in G' then $f^{-1}(H)$ is normal in G. (6)
   iv. Show that $\{e, b\}$ is normal in $\{e, b, a^2b, a^2\}$ but not normal in $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = e = b^2, aba = b$. (6)

2. (a) Answer any **ONE**
   i. Show that finite integral domain is a field. Give example of infinite integral domain that is not a field. Give example of a finite ring that is not an integral domain. (8)
   ii. Define characteristic of ring. Let 1_R be the multiplicative identity of R. Show that R has characteristic n if and only if order of 1_R in the group $(R, +)$ is n. Hence or otherwise show that the characteristic of integral domain is either zero or prime. (8)

   (b) Answer any **TWO**
   i. Let R, R' be commutative rings and $f : R \to R'$ be an onto homomorphism. If I is an ideal of R, show that $f(I)$ is an ideal of R'. (6)
   ii. Show that the only ideals in a field F are zero ideal and F itself. (6)
   iii. Find the kernel of the ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi(f(x)) = f(2+i)$. Find $f(x) \in \mathbb{R}[x]$ such that ker $\phi = (f(x))$. (6)
   iv. Show that a field containing 8 elements has characteristic 2. (6)

3. (a) Answer any **ONE**
   i. Show that any prime element of an integral domain is also an irreducible element. Further show that the converse is true in PID. (8)
   ii. Define an Euclidean domain. Show that every Euclidean domain is a PID. (8)

[P.T.O.]

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(b) Answer any **TWO**

i. Show that a nonzero ideal P of a commutative ring R is prime if and only if R/P is an integral domain.

ii. Show that the only maximal ideals in $\mathbb{C}[x]$ are $(x-\alpha)$ for $\alpha \in \mathbb{C}$.

iii. Show that $\{f(x) \in \mathbb{Z}[x] \ / \ 2 \mid f(0)\}$ is not a principal ideal in $\mathbb{Z}[x]$.

iv. Check whether x^2+1 and x^2+x+5 are irreducible in $\mathbb{Z}_{11}[x]$.

4. Answer any **THREE**

(a) Find a subgroup of S_4 isomorphic to $\mathbb{Z}_4$.

(b) Find the order of each element in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(c) Check whether $\left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \ / \ a \in \mathbb{R}\right\}$ is an ideal of the ring $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ / \ a, b, d \in \mathbb{R}\right\}$.

(d) Consider the ring $R = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} \ / \ a, b \in \mathbb{R}\right\}$. Show that the map $\phi : R \to \mathbb{Z}$ defined by $\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = a - b$ is a ring homomorphism. Also find the kernel.

(e) Show that an ideal $m\mathbb{Z}$ is maximal in $\mathbb{Z}$ if and only if m is prime.

(f) Show that $Q[x]/(x^3-2)$ is a field.

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