

Duration: $2\frac{1}{2}$ Hours

OLD COURSE

Max. Marks : 75

1) All questions are compulsory

2) Figures to the right indicate marks.

Q.1 (a) Attempt any ONE of the following (8)

- (i) If $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ and if $\lim_{x \rightarrow a} f(x,y)$ and $\lim_{y \rightarrow b} f(x,y)$ both exists, then prove that

$$\lim_{x \rightarrow a} (\lim_{y \rightarrow b} f(x,y)) = \lim_{y \rightarrow b} (\lim_{x \rightarrow a} f(x,y)) = L.$$

Give an example to show that the converse is not true.

- (ii) Let S be an open subset of \mathbb{R}^n and $f, g : S \rightarrow \mathbb{R}^m$ and let $a \in S$, $\lambda \in \mathbb{R}$. If $\lim_{x \rightarrow a} f(x) = b$ and $\lim_{x \rightarrow a} g(x) = c$ then using $\epsilon - \delta$ definition prove that p) $\lim_{x \rightarrow a} (\lambda f(x)) = \lambda b$ q) $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = b \cdot c$

(b) Attempt any TWO of the following (12)

- (i) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f(x, y) = (x^2 - y^2, x^2 + y^2)$. Using $\epsilon - \delta$ definition show that each component of f is continuous at $(1, 2)$.
- (ii) If $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = |x| + |y|$ then show that $f_x(0,0)$ and $f_y(0,0)$ do not exist. Check whether f is continuous at $(0, 0)$
- (iii) Find $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{\sqrt{(x^2+y^2)}}$ using polar co-ordinates.
- (iv) Using $\epsilon - \delta$ definition, discuss the continuity of $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $f(x, y) = (x^2 + 2y, y^2 + 2x)$ at $(2, 3)$

Q.2 (a) Attempt any ONE of the following (8)

- (i) Let S be an open subset of \mathbb{R}^2 and $f: S \rightarrow \mathbb{R}$ be such that $D_1f, D_2f, D_{12}f, D_{21}f$ exists on S . If $(a, b) \in S$ and $D_{12}f, D_{21}f$ are continuous on S , then show that $D_{12}f(a, b) = D_{21}f(a, b)$.
- (ii) Let S be an open subset of \mathbb{R}^n and $f: S \rightarrow \mathbb{R}$ be differentiable at $a \in S$ with total derivative $Df(a)$. Show that $f'(a; y)$ exists for all $y \in \mathbb{R}^n$ and $f'(a; y) = Df(a)$ and $f'(a; y) = \sum_{k=1}^n D_k f(a) y_k = \nabla f(a) \cdot y$ for all $y \in \mathbb{R}^n$.

(b) Attempt any TWO of the following (12)

- (i) State and prove the Mean Value Theorem for a scalar field
- (ii) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable function. Let $A(1,3), B(3,3), C(1,7), D(6,15)$. The directional derivative of f at A in the direction of AB is 3 and in the direction of AC is 26. Find the directional derivative of f at A in the direction of AD .
- (iii) If $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, g: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ $f(x, y) = (xy^2, xy), g(u, v) = (u + v, u - v, uv)$.
 (p) Compute the Jacobian matrices $Df(x, y), Dg(u, v)$ & $D(g \circ f)(x, y)$
 (q) Verify that $D(g \circ f)(1, 1) = Dg(1, 1) \cdot Df(1, 1)$
- (iv) Let $z = e^{u+v+w}$, where $u = x^2 \sin^2 y, v = 2x \sin x \sin y, w = y^2$. Use chain rule to find z_x, z_y .

Q.3 (a) Attempt any ONE of the following (8)

- (i) State and prove Stoke's Theorem for an oriented smooth simple parametrized surface in \mathbb{R}^3 bounded by simple, closed, curve traversed counter clockwise assuming general form of Green's Theorem.
- (ii) State Divergence Theorem for a solid in 3 – space bounded by an orientable closed surface with positive orientation and prove the Divergence theorem for cubical region.

(P.T.O)

(b) Attempt any TWO of the following**(12)**

- (i) Compute the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$
- (ii) Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n} dS$ if $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the sphere $x^2 + y^2 + z^2 = 1$.
- (iii) Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \vec{F} \cdot \hat{n} ds$, where $\vec{F}(x, y, z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ above the xy -plane.
- (iv) Verify Divergence Theorem for vector field $\vec{F}(x, y, z) = 3x\hat{i} + xy\hat{j} + 2xz\hat{k}$ and V is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Q.4 Attempt any THREE of the following**(15)**

- (i) If $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, then find $\lim_{x \rightarrow 0}(\lim_{y \rightarrow 0} f(x, y))$ and $\lim_{y \rightarrow 0}(\lim_{x \rightarrow 0} f(x, y))$. Also find $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ if exists

- (ii) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right), & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}$$

Find $D_{12}f(0, 0), D_{21}f(0, 0)$ and check whether they are equal.

- (iii) Find all differentiable vector fields $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for which the Jacobian matrix $Df(x, y, z) = \text{diag}(p(x), q(y), r(z))$ where $p, q, r: \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.
- (iv) Given $u = f(x, y)$ has continuous partial derivatives with respect to x and y . If $x = r \cos \theta, y = r \sin \theta$, then show that $u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2} u_\theta^2$.
- (v) Evaluate the surface integral $\iint_S xz dS$, where S is the triangle with the vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.
- (vi) Define the Fundamental Vector Product for a surface S whose vector equation is $\vec{r}(u, v) = X(u, v)\hat{i} + Y(u, v)\hat{j} + Z(u, v)\hat{k} \quad \forall (u, v) \in T$ in uv -plane. Compute $\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|$ for $\vec{r}(u, v) = v \sin \alpha \cos u \hat{i} + v \sin \alpha \sin u \hat{j} + v \cos \alpha \hat{k}$