Duration: 2 $\frac{1}{2}$Hrs Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) Show that in a metric space (X, d)
      (I) an arbitrary union of open sets is an open set.
      (II) a finite intersection of open sets is an open set.
   (ii) Define an open ball $B(x, r)$ in a metric space (X, d) and show that every open ball is an open set. Also give an example to show that the converse need not be true.

   (b) Attempt any Two from the following: (12)
   (i) Define a normed linear space $(X, \| \ \|)$. Show that in a normed linear space $(X, \| \ \|)$, $B(x, r) = x + rB(0, 1), \ \forall x \in X, \ r > 0$.
   (ii) Define distance of a point p from a set A in a metric space (X, d). If $A \subseteq X$ then show that $|d(x, A) - d(y, A)| \leq d(x, y), \forall x, y \in X$.
   (iii) d_1, d_2 are two metrics on $\mathbb{R}^2$ defined by

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$$
$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

   where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Show that d_1 and d_2 are equivalent metrics.
   (iv) State and prove the Hausdorff property in a metric space (X, d).

2. (a) Attempt any One from the following: (8)
   (i) Show that for a subset F of a metric space (X, d), F is closed if and only if F contains all its limit points.
   (ii) (X, d) is a metric space and $A, B \subseteq X$. Then prove the following:
      (I) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$
      (II) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and equality may not hold.
      (III) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (b) Attempt any Two from the following: (12)
   (i) Let (X, d) be a metric space. If sequence (x_n) is a Cauchy sequence in (X, d) and the sequence (x_n) has a convergent subsequence which converges to $p \in X$, then prove that the sequence (x_n) also converges to p.
   (ii) Show that for any subset A of a metric space (X, d), $\delta(A) = \delta(\overline{A})$ where δA indicates the diameter of A.
   (iii) Let d_1 and d_2 be metrics on a non-empty set X such that there exists $k_1, k_2 > 0$ such that $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y) \ \forall x, y \in X$ then show that a sequence (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2).
   (iv) Which of the following are dense subsets of $\mathbb{R}$ with the usual distance? Justify your answer.
      (I) $\mathbb{Q}$ (II) $\mathbb{Z}$ (III) $\mathbb{R} \backslash \mathbb{Z}$

3. (a) Attempt any One from the following: (8)
   (i) Show that if a subset K of $\mathbb{R}^n$ is sequentially compact then it is closed and bounded. (distance being Euclidean)
   (ii) Show that a subset K of $\mathbb{R}^n$ has the Bolzano-Weierstrass property if and only if K is sequentially compact. (distance being Euclidean)

   (b) Attempt any two from the following: (12)
   (i) If A, B are compact subsets of $\mathbb{R}$ then show that $A+B$ is also a compact subset of $\mathbb{R}$. (distance being usual)
   (ii) Prove that a subset of a discrete metric space is compact if and only if it is finite.
   (iii) Show that a closed subset of a compact metric space is compact.
   (iv) Consider the metric space $(\mathbb{R}, d)$, where d is the usual distance. Show that $\{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ is an open cover of $(0,1)$. Is $(0,1)$ a compact set? Justify your answer.

4. Attempt any Three from the following: (15)
   (a) Prove or disprove: Every open ball in $(\mathbb{N}, d_1)$ is an open ball in $(\mathbb{N}, d)$ where d_1 is the discrete metric on $\mathbb{N}$ and d is the usual metric.
   (b) Show that $U = \{(x, y) \in \mathbb{R}^2 : \ 2x - 3y < 1\}$ is an open subset of $\mathbb{R}^2$ with the Euclidean metric.
   (c) Consider the sequence (f_n) of functions in $C[0,1]$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \dfrac{n}{2} + 1 & \text{if} \quad \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < t \le 1 \end{cases}$$

   Show that $\{f_n\}$ is Cauchy w.r.t. $\| \ \|_1$ where $\|f\|_1 = \displaystyle\int_0^1 |f(t)|dt$.
   (d) If (x_n) and (y_n) are sequences in a metric space (X, d) such that $x_n \longrightarrow p$ and $y_n \longrightarrow q$ then show that the sequence of real numbers $d(x_n, y_n) \longrightarrow d(p, q)$ in $(\mathbb{R}, \text{usual})$.
   (e) Prove or disprove:
   (I) Interior of a compact set is compact.
   (II) Every compact subset of $(\mathbb{R}, \text{usual})$ has a limit point.
   (f) (X, d) be a compact metric space. If $\{A_n\}$ is a sequence of non-empty closed sets in X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ then show that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.