

(3 Hours)

[Total Marks: 100]

- N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following:

(20)

- i. The integral $\int_0^2 \int_x^{x\sqrt{3}} f(\sqrt{x^2 + y^2}) dy dx$ in polar coordinates is
- (a) $\int_0^{\pi/4} \int_0^{2\sec\theta} f(r) r dr d\theta$ (b) $\int_0^{\pi/3} \int_0^{2\sec\theta} f(r) r dr d\theta$
- (c) $\int_{\pi/4}^{\pi/3} \int_0^{2\sec\theta} f(r) r dr d\theta$ (d) $\int_{\pi/4}^{\pi/3} \int_0^{2\cos\theta} f(r) r dr d\theta$
- ii. $I = \int_0^1 \int_{x^2}^x x f(y) dy dx$ where f is continuous function defined on $[0,1]$. Then I is
- (a) $\frac{1}{2} \int_0^1 (y - y^2) f(y) dy$ (b) Independent of $f(y)$
- (c) $\frac{1}{2} \int_0^1 (y^2 - y) f(y) dy$ (d) $f(x)$
- iii. The volume of region bounded by $z = x + y, z = 6, x = 0, y = 0, z = 0$ is
- (a) 36 cubic units (b) 30 cubic units
- (c) 216 cubic units (d) None of these.
- iv. The parametric equations $x = \cos(\cos t), y = \sin(\cos t), t \in [0, \pi]$ describes
- (a) one full circle
- (b) an arc of a circle in first quadrant
- (c) an arc of a circle in the first and fourth quadrant
- (d) None of the above.
- v. The line integral $\int_C \vec{F} \cdot d\vec{r}$; $\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$ and $C: x^2 + y^2 = a^2$.
- (a) depends on a
- (b) does not exist as Green's Theorem is not applicable
- (c) is a constant independent of ' a '
- (d) None of the above
- vi. $\nabla f(x, y, z) = 2xyze^{x^2}\hat{i} + ze^{x^2}\hat{j} + ye^{x^2}\hat{k}$ and $f(0,0,0) = 7$. Then $f(1,1,2) = ?$
- (a) $2e + 5$ (b) $2e + 7$
- (c) $e + 5$ (d) $7e + 2$
- vii. The magnitude of the fundamental vector product $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ for surface $\vec{r}(u, v) = (u + v)\hat{i} + (u - v)\hat{j} + 4k$ is
- (a) $\sqrt{4 + v^2}$ (b) $\sqrt{4 + 128v^2}$
- (c) $\sqrt{4v^2 + 1}$ (d) None of these

- viii. The flux of the vector field $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$ equals
- (a) $\frac{4}{3}\pi$ (b) $\frac{2}{3}\pi$
 (c) $\frac{1}{3}\pi$ (d) None of these
- ix. The surface integral $\iint_S (\vec{r} \cdot \hat{n})dS$ over a closed surface S with volume V , where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is
- (a) V (b) $3V$
 (c) 0 (d) None of these
- x. $\text{div}(\text{curl}(x^2, yz, \sin z))$ is
- (a) $2x + z + \cos z$ (b) 0
 (c) $\bar{0}$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

- i. Define the double integral of a bounded function $f: S \rightarrow \mathbb{R}$ where $S = [a, b] \times [c, d]$ is a rectangular region in $\mathbb{R}^2$ and using usual notation show that $m(b-a)(d-c) \leq \iint_S f \leq M(b-a)(d-c)$.
- ii. State and prove Fubini's Theorem for a rectangular domain in $\mathbb{R}^2$.

b) Attempt any TWO questions from the following: (12)

- i. Prove that every continuous function defined on a rectangular domain D in $\mathbb{R}^2$ is integrable.
- ii. Evaluate the integral $\iint_S 3y dA$ using polar coordinates, where S is the region in the first quadrant bounded above by the circle $(x-1)^2 + y^2 = 1$ and below by the line $y = x$.
- iii. Using cylindrical co-ordinates find the volume of the solid region S in $\mathbb{R}^3$ which is bounded by the paraboloid $x^2 + y^2 = 4 - z$ and the plane $z = 0$.
- iv. Evaluate $\iiint_S \frac{dx dy dz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$ where S is region in $\mathbb{R}^3$ between the two spheres with centre at the origin and radii 2 and 5.

Q.3 a) Attempt any ONE question from the following: (08)

- i. Let f be a continuously differentiable scalar field defined on an open set U in $\mathbb{R}^n$. Suppose P, Q are two points of U that can be connected by piecewise smooth curve C lying in U . Prove that $\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$ given that C has parameterization $\vec{r}(t)$, $t \in [a, b]$ with $\vec{r}(a) = P$ and $\vec{r}(b) = Q$. Further if $F = \nabla f$ where $(x, y) = \cos(x + 2y)$, does there exist a smooth, closed path C such that $\int_C F \cdot d\vec{r} = \pi$? If so, find such a path C .

- ii. State and prove Green's Theorem for a rectangle. Evaluate

$$\oint_C (3y - e^{\cos x})dx + (7x + \sqrt{y^5 + 1})dy \text{ where } C \text{ is the circle } x^2 + y^2 = 9.$$

- b) Attempt any TWO

(12)

- Show that the vector $F = (y \sin z, x \sin z, xy \cos z)$ is conservative. If so find f such that $\nabla f = F$.
- Using Green's Theorem, find the area of the region D whose boundary is positively oriented simple closed curve bounded by the lines $y = 1, y = 3, x = 0$ and the parabola $y^2 = x$.
- Show that two equivalent parameterized curves in $\mathbb{R}^n$ have essentially the same image set. Show that the converse is not true by considering curves $\alpha(t) = (\cos t, \sin t)$ and $\beta(t) = (\sin t, \cos t)$, $0 \leq t \leq 2\pi$.
- $F = (P, Q)$ is a continuously differentiable function defined on a simply connected region D in $\mathbb{R}^2$. Show that $\int_C Pdx + Qdy = 0$ around every piecewise smooth closed curve C in D if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \forall (x, y) \in D$.

- Q.4 a) Attempt any ONE.

(08)

- Let $S = \vec{r}(T)$ be a smooth parametric surface described by a differentiable function $\vec{r}$ defined on region T . Let f be defined and bounded on S . Define surface integral of f over S . If R and r are smoothly equivalent functions, $R(s, t) = \vec{r}(G(s, t))$ where $G(s, t) = u(s, t)\hat{i} + v(s, t)\hat{j}$ being continuously differentiable. Then show that $\iint_{r(A)} f dS = \iint_{R(B)} f dS$ where $G(B) = A$.
- State Divergence Theorem for a solid in 3-space (or $\mathbb{R}^3$) bounded by an orientable closed surface with positive orientation and prove the divergence Theorem for cubical region.

- b) Attempt any TWO.

(12)

- Let $S = \vec{r}(T)$ be a smooth parametric surface in uv plane. Define area of S . If S is represented by an equation $z = f(x, y)$ then show that area of S is given by $\iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy$ where T is projection of S on XY -plane.
- Evaluate the surface integral of $F(x, y, z) = (x, y, 0)$ over S , where S is the hemisphere above XY -plane of radius 2.
- Find surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

- iv. Use Stokes' theorem to evaluate $\iint_S (\text{curl } F) \cdot n dS$ where $F(x, y, z) = y\hat{i} + x\hat{j} + xz\hat{k}$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1$; $z \geq 0$ and n is the unit normal with a non-negative z component.

Q.5 Attempt any FOUR.

(20)

- Using double integration, find the area of the region S in $\mathbb{R}^2$ bounded by the parabola $y = 9 - x^2$ and $y = x^2 + 1$.
- Evaluate $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy$
- Let U be an open set in $\mathbb{R}^n$ and $\alpha : [a, b] \rightarrow U$ be a parameterization of curve Γ . If $f, g : U \rightarrow \mathbb{R}$ are continuous functions, then prove that $\int_{\Gamma} (cf + dg) = c \int_{\Gamma} f + d \int_{\Gamma} g$, where c, d are real constants.
- Evaluate the integral of the vector field, $F(x, y, z) = (x^2 - xy, 1)$ along the circle of radius 1, with centre at the origin and lying in the yz , plane, traversed counterclockwise as viewed from the positive x axis.
- Use Gauss Divergence theorem to evaluate $\iint_S F \cdot n dS$ where $F(x, y, z) = (x + y, y + z, z + x)$ and S is the region given by $-4 \leq z \leq 4$; $0 \leq x^2 + y^2 \leq 4$.
- Evaluate the surface integral of $f(x, y, z) = z$ over the surface parameterized by $\alpha(u, v) = (u \cos v, u \sin v, 1)$; $1 \leq u \leq 2, 0 \leq v \leq 2\pi$
