**Q. P. Code : 20499**

**Time : 2:30 hours** **Marks : 75**

**N.B. 1.** All questions are compulsory.
**2.** From Question 1,2 and 3, Attempt any one from part(a) and any two from part(b).
**3.** From Question 4, Attempt any THREE
**4.** Figures to the right indicate marks for the respective parts.

Q1 a i Prove that a continuous function is integrable on a rectangular domain. 8

ii State the Change of variable formula for a triple integral stating the conditions under which it is valid. Explain further , how will you use it to a express triple integral in spherical co-ordinates ( $\rho, \theta, \varphi$).

b i Use Fubini's Theorem to evaluate $\iint_S f$ where $f(x,y) = x + y$ and $S$ is defined by $S = \{(x\,,y) : |x| \le 1\,, 0 \le y \le 1 + |x|\}$. 12

ii Use polar co-ordinates to find area of a region $S$ in the first quadrant of circle $x^2 + y^2 - 8y = 0$ below the line $y = \sqrt{3}x$.

iii Using cylindrical co-ordinates, find the volume of solid $S$ bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

iv State Leibnitz rule for differentiation under integral sign. Hence find $g'(x)$ for $g(x) = \int_0^1 \log(x^2 + y^2)\, dy$. Verify your answer by finding $g'(x)$ by direct method.

Q2 a i Let $f$ be a continuously differentiable scalar field defined on an open set $U$ in $\mathbb{R}^n$. Suppose $P, Q$ are two points of $U$ that can be connected by piecewise smooth curve $C$ lying in $U$. Prove that $\int_C \nabla f \cdot dr = f(Q) - f(P)$ given that $C$ has parameterization $r(t)$, $t \in [a,b]$ with $r(a) = P$ and $r(b) = Q$. 8
Further if $F = \nabla f$ where $f(x,y) = sin(x - 2y)$ , does there exist a smooth, closed path $C$ such that $\int_C F.dr = 1$ ? If so, find such a path $C$.

ii State and prove Green's Theorem for a rectangle.
Evaluate $\oint_c \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4 + 1}\,\right) dy$ where $C$ is the circle $x^2 + y^2 = 9$.

b i Define the line integral of a vector field $F$ defined on an open set $U$ in $\mathbb{R}^n$ along an oriented curve $\Gamma$ in $U$. If $\Gamma$ and $\Gamma'$ are two equivalent but orientation reversing curves in $U$, show that $\int_\Gamma F = -\int_{\Gamma'} F$. 12

ii Evaluate $\int_C y^2 dx + x dy$ where
(1) $C$ is the line segment from $(-5,-3)$ to $(0,2)$
(2) $C$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$
(3) Is $F = (y^2, x)$ conservative? Justify your answer.

iii Evaluate the line integral $\int_{(0,1)}^{(1,2)} (1 - ye^{-x})dx + (e^{-x})\, dy$.

iv A force field $F(x,y) = cxy\, i + x^6 y^2 j$, ('$c$' is a positive constant) acts on a particle which moves it from (0,0) to the line $x = 1$ along a curve of the form $y = ax^b$ where $a > 0\,, b > 0$. Find a value of '$a$' (in terms of '$c$') if the work done by this force is independent of $b$.

**Q. P. Code : 20499**

Q3 a i State and prove Stokes' theorem for an oriented smooth, simple parameterized surface in $\mathbb{R}^3$ bounded by a simple, closed curve traversed counter clockwise. 8

ii Let $S$ and $V$ satisfy hypothesis of Divergence Theorem, scalar fields $f, g$ have continuous second order partial derivatives and $n$ is the unit outward normal vector to surface $S$. For $\bar{r} = x\,i + y\,j + z\,k$ and $r = ||\bar{r}||$. Prove the following.

(1) $\int\int_S (r^n\,\bar{r}).\,dS = (n+3)\int\int\int_V r^n\,dV$

(2) $|V| = \frac{1}{3}\int\int_S \bar{r}.n\,dS$ where $|V|$ is volume of $V$.

(3) $\int\int_S (f\,\nabla g).n\,dS = \int\int\int_V (f\,\nabla^2 g + \nabla f.\nabla g)\,dV$

b i Evaluate $\int\int_S F.ndS$ where $S$ is the hemisphere above the $XY$ plane with unit radius and $F(x,y,z) = (x,y,0)$. 12

ii Let $S = r(T)$ be a smooth parametric surface described by a differentiable function $r$ defined on a region $T$. Let $f$ be a scalar field and bounded on $S$. If $R$ and $r$ are smoothly equivalent parametrizations with $R(s,t) = r(G(s,t))$ where $G(s,t) = u(s,t)i + v(s,t)\,j$ is a one to one continuously differentiable map, then show that $\iint_{r(A)} f ds = \iint_{R(B)} f ds$ where $G(B) = A$.

iii Use Gauss Divergence theorem to find $\int\int_S F.ndS$ where $F(x,y,z) = (x+y, y+z, z+x)$ and $S$ is the region given by $-4 + x^2 + y^2 \le z \le 4 - x^2 - y^2$.

iv Use Stokes' theorem to evaluate $\int_C x^4 dx - xy\,dy + z^2\,dz$ where $C$ is the boundary of the tetrahedron with vertices $(2,0,0), (0,2,0)$ and $(0,0,2)$ lying in the first octant.

Q4 i Use spherical co-ordinates to evaluate $\iiint_S z\,dxdydz$ where $S$ is the solid enclosed by $x^2 + y^2 + z^2 = 1, z \ge 0$. 15

ii Let $D = \{(x,y): a \le x \le b, -\emptyset(x) \le y \le \emptyset(x)\}$ where $\emptyset$ is a non negative continuous function on $[a,b]$. Let $f(x,y)$ be a function on $D$ such that $f(x,y) = f(x,-y)$ $\forall\,(x,y) \in D$. Show that $\iint_D f\,dA = 0$.

iii Let $U$ be an open set in $\mathbb{R}^n$ and $\alpha : [a,b] \to U$ be a parameterization of curve $\Gamma$. If $f : U \to \mathbb{R}$ is a continuous function, then show that $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$, where $\Gamma_1$ and $\Gamma_2$ are restrictions of $\alpha$ to $[a,c]$ and $[c,d]$ where $a < c < b$.

iv Evaluate the line integral of $f(x,y,z) = x + y + z$, along the path $\gamma(t) = (\sin t, \cos t, t),\ 0 \le t \le 2\pi$.

v Find surface area of $S$, where $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 9$.

vi If $S$ and $C$ satisfy hypothesis of Stokes' Theorem and $f, g$ have continuous second order partial derivatives. Then prove that $\int_C (f\nabla g).dr = \int\int_S (\nabla f \times \nabla g).n\,dS$

*******************