

3 Hours]

[Total Marks: 100

N.B.: (1) All questions are compulsory.

(2) Figures to the right indicate marks for respective subquestions.

1. Fill in the blank by choosing the correct option.

(i) Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ and $V = \{B \in M_4(\mathbb{R}) : AB = BA\}$. Then, (2)

(a) $\dim V = 4$ and $\dim M_4(\mathbb{R})/V = 12$

(b) $\dim V = 8$ and $\dim M_4(\mathbb{R})/V = 8$

(c) $\dim V = 8$ and $\dim M_4(\mathbb{R})/V = 16$

(d) None of these.

(ii) Let α be an orthogonal transformation of the plane such that the (2)

matrix of α w. r. t. the standard basis of $\mathbb{R}^2$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$,

then α represents a

(a) rotation about origin through $\frac{\pi}{4}$.

(b) rotation about origin through $\frac{5\pi}{4}$.

(c) rotation about the line $y = -x$. (d) None of the above.

(iii) A 2×2 matrix A has the characteristic polynomial $x^2 + 2x - 1$, (2)
then the value of $\det(2I_2 + A)$ is

(a) $\frac{1}{\det A}$ (b) 0 (c) $2 + \det A$ (d) $2 \det A$

(iv) Let A and B be square matrices such that $AB = I$ then zero is an (2)
eigenvalue of

(a) A but not B (b) B but not A

(c) both A and B (d) neither A nor B .

(v) If λ is a characteristic root of a matrix A then characteristic roots (2)
of $-A$ and $\alpha I - A$ respectively are

(a) $-\lambda$ and $\alpha - \lambda$ (b) $-\lambda$ and α

(c) $-\lambda$ and λ (d) None of these.

(vi) Which of the following statements are true (2)

(p) If the characteristic roots of two $n \times n$ matrices are same then their characteristic polynomials are same.

(q) If the characteristic polynomials of two $n \times n$ matrices are same then their characteristic roots are same.

- (r) If eigen values of two $n \times n$ matrices are same then their eigen vectors are same.
- (s) The characteristic roots of two $n \times n$ matrices are same but their characteristic polynomials may not be same.
- (a) (q) and (s) are true. (b) (p), (r) are true.
- (c) (p), (q) and (r) are true. (d) only (q) is true.
- (vii) The minimal polynomial of the diagonal matrix $A = \text{diag} \{1, -1, 1, -1\}$ is $x^2 + 1$ (a) $x^2 + 1$ (b) $x^2 - 1$ (c) $(x^2 - 1)^2$ (d) None of these. (2)
- (viii) Let $A = \begin{bmatrix} 0 & a \\ 0 & -a \end{bmatrix}$ (a) A is orthogonally diagonalizable if and only if $a = 1$ (b) A is not diagonalizable for any $a \in \mathbb{R}$. (c) A is diagonalizable but not orthogonally diagonalizable. (d) None of these. (2)
- (ix) If $A, B, C, D \in M_2(\mathbb{R})$ such that A, B, C, D are non-zero and not diagonal. If $A^2 = I, B^2 = B, C^2 = 0, C \neq 0$ and every eigenvalue of D is 2, then (a) A, B, C, D are all diagonalizable. (b) B, C, D are diagonalizable. (c) A, B are diagonalizable (d) Only D is diagonalizable. (2)
- (x) The quadratic form $Q(x) = x_1^2 + 4x_1x_2 + x_2^2$ has (a) rank = 1, signature = 1. (b) rank = 2, signature = 0. (c) rank = 2, signature = 2. (d) None of the above. (2)
2. (a) Answer any **ONE**
- (i) Let V be a finite dimensional inner product vector space and $T : V \rightarrow V$ be a linear transformation. Prove that the following statements are equivalent. (8)
- (p) T is orthogonal.
- (q) $\|T(X)\| = \|X\|$ for all $X \in V$.
- (r) If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V , then $\{T(e_i)\}_{i=1}^n$ is also an orthonormal basis of V .
- (ii) State and prove the Cayley Hamilton Theorem. (8)
- (b) Answer any **TWO**
- (i) State and prove the 'First Isomorphism Theorem of vector space' (Fundamental theorem of vector space homomorphism). (6)
- (ii) Let $(V, \langle \rangle)$ be an n dimensional inner product space and W be a subspace of V of dimension $n - 1$. Let u be a unit (6)

vector orthogonal to W . Show that $T : V \rightarrow V$ defined by $T(x) = x - 2\langle x, u \rangle u$ is an orthogonal linear transformation such that $T(w) = w, \forall w \in W$ and $T(u) = -u$.

(iii) Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \end{pmatrix}$. A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is (6)

defined by $T(x) = AX$ (X being a column vector in $\mathbb{R}^3$). Find $\ker T$, a basis of $\ker T$ and $\mathbb{R}^3 / \ker T$.

(iv) Show that $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $\alpha(x, y, z) = (\frac{x}{2} + \frac{\sqrt{3}}{2}z - 1, y, \frac{\sqrt{3}}{2}x - \frac{z}{2} + 5)$ is an isometry. Express it as a composite of an orthogonal transformation and a translation. (6)

3. (a) Answer any **ONE**

(i) Define eigen value of a real square matrix. Show that, if λ is an eigen value of a real $n \times n$ matrix A , then (8)

(p) λ is an eigen value of A^t .

(q) λ^k is an eigen value of A^k for $k \in \mathbb{N}$. Hence $f(\lambda)$ is an eigen value of $f(A)$, for a polynomial $f(x)$ over $\mathbb{R}$.

(r) If A is invertible, then λ^{-1} is an eigen value of A^{-1} .

(ii) Show that minimal polynomial of a real matrix $A_{n \times n}$ divides every polynomial which annihilates A . Further show that λ is a root of the minimal polynomial of matrix A if and only if λ is a characteristic root of A . (8)

(b) Answer any **TWO**

(i) If A is an $n \times n$ real matrix, and $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigen value of A with $X_1, X_2, \dots, X_k$ as corresponding eigenvectors, then show that $X_1, X_2, \dots, X_k$ are linearly independent. (6)

(ii) Let $A_{n \times n}$ be a real matrix. if A has n distinct characteristic roots, then prove that the characteristic polynomial of A = the minimal polynomial of A . (6)

(iii) Find the eigen values and eigen vectors of $A_{2 \times 2}$ such that (6)

$$\sum_{j=1}^2 a_{ij} = 1 \text{ for } i = 1, 2.$$

(iv) Let v be a non-zero vector in $\mathbb{R}^n$ and $A = vv^T$ where v is treated as a $n \times 1$ column vector. Find the minimal polynomial of A . (6)

4. (a) Answer any **ONE**

(i) Define an orthogonally diagonalizable matrix. Show that every real symmetric matrix is orthogonally diagonalizable. (8)

- (ii) Let A be real symmetric matrix of order n . Show that characteristic roots of A are real. Also show that if λ_1, λ_2 are distinct eigen values of A and X_1, X_2 are corresponding eigen vectors then X_1, X_2 are orthogonal. (8)
- (b) Answer any **TWO**
- (i) Show that an $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigen vectors of A . (6)
- (ii) Show that every quadratic form $Q(x_1, x_2, \dots, x_n)$ over $\mathbb{R}$ can be reduced to standard form $\sum_{i=1}^n \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$, where X, Y are column vectors in $\mathbb{R}$ and P is an $n \times n$ orthogonal matrix. (6)
- (iii) For $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix. Hence or otherwise find A^{1000} and B such that $B^3 = A$. (6)
- (iv) Let A be a 13×13 real matrix of rank 1. Find the eigen values and show that geometric multiplicity is equal to algebraic multiplicity for each eigen value. (6)

5. Answer any **FOUR**

- (a) Find the orthogonal transformations in $\mathbb{R}^3$ which represent reflections with respect to $2x - y + z = 0$. (5)
- (b) Express the characteristic polynomial of $aI + bA$ in terms of the characteristic polynomial of A . (5)
- (c) Prove or disprove: Two matrices are similar if and only if their characteristic polynomials are same. (5)
- (d) Define invariant subspace with respect to a linear transformation. Check which of $V = \{(x, 0) : x \in \mathbb{R}\}$ and $W = \{(x, -x) : x \in \mathbb{R}\}$ are $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ invariant where $T(x, y) = (y, x)$. Also identify the eigen value of T with respect to which W is an eigen space. (5)
- (e) If $A^2 = A$ for non-zero $n \times n$ matrix A then show that algebraic multiplicity of eigen value 1 is rank A . (5)
- (f) Find value of k , for which the symmetric matrix associated to the quadratic form $3x_1^2 + x_2^2 + 2x^2 + 2x_1x_3 + 2kx_2x_3$ is positive definite. State the result used. (5)
