Duration:[3 Hours] **[Total Marks: 100]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. The smallest n such that the complete graph K_n has atleast 600 edges.
(a) 35 (b) 36
(c) 40 (d) 37

ii. Every vertex induced subgraph of a complete graph
(a) is complete (b) bipartite
(c) disconnected (d) acyclic

iii. Which one of the following sequences is graphic?
(a) 2, 3, 3, 3, 3, 3, 4, 5 (b) 2, 3, 3, 3, 4, 7
(c) 1, 2, 3, 4, 5, 6 (d) 2, 3, 3, 3, 3, 3, 4

iv. If $A(G)$ is adjacency matrix of graph G then number of $1's$ in each row or column denotes
(a) Number of edges in graph G
(b) Degree of corresponding vertex
(c) Degree of the corresponding vertex inG^c
(d) None of the above.

v. If there is a tree with 3 vertices of degree 2, 4 vertices of degree 3 and 3 vertices of degree 4 then the number of pendent vertices in tree is
(a) 12 (b) 2 (c) 11 (d) 13

vi. The number of different labeled trees of order 15 is:
(a) 15^2 (b) 15^{13}
(c) 13^{15} (d) None of the above.

vii. How many edges does a full binary tree with 1000 internal vertices have?
(a) 2001 (b) 2000
(c) 1000 (d) 999

viii. A connected graph has Eulerian trail if it has
(a) At most two vertices of odd degree
(b) Exactly two vertices of odd degree
(c) At least two vertices of odd degree
(d) None of these

ix. If G is Hamiltonian and if S is any non empty proper subset of $V(G)$ then
(a) $\omega(G-S) = |S|$ (b) $\omega(G-S) \leq |S|$
(c) $\omega(G-S) \geq |S|$ (d) None of these

x. Number of edges in Q_4 is
(a) 24 (b) 16 (c) 18 (d) 32

2. (a) Attempt any **ONE** question from the following: (8)

   i. If $(A^n) = (a^n_{ij})$ is the n^{th} power of adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, \ldots v_n\}$, then prove that

      (a) a^2_{ij}, $i \neq j$ is the number of $v_i - v_j$ path of length 2.

      (b) $a^2_{ii} = deg(v_i)$

      (c) $\frac{1}{6}$trace of A^3 is the number of triangles in G.

   ii. Define a self complementary graph. If G is self complementary graph of order p, show that G is connected and $p \equiv 0$ or $1(\mod 4)$.

   (b) Attempt any **TWO** questions from the following: (12)

   i. State $Havel - Hakimi$ theorem for degree sequence of a graph. Check whether the sequence $5, 4, 3, 3, 2, 2, 2, 1, 1, 1$ is graphical or not? If graphical, construct a graph, for which the given sequence is a degree sequence of the graph. If not, Justify you answer.

   ii. Show that the number of edges of a simple graph with p vertices and k components cannot exceed $\frac{(p-k)(p-k+1)}{2}$.

   iii. Let G be a simple graph and $\delta(G) \geq 2$, then show that there exists a cycle of length at least $\delta(G) + 1$ in G.

   iv. Prove that every (p, q) graph with $q \geq p$ contains a cycle. Is it true if $q \geq p - 1$? Justify.

3. (a) Attempt any **ONE** question from the following: (8)

   i. Define a spanning tree of a graph G. Show that a graph G is connected if and only if it has a spanning tree.

   ii. State and prove Cayley's formula for spanning trees.

   (b) Attempt any **TWO** questions from the following: (12)

   i. Define a cut vertex of a graph G. Show that vertex v is a cut vertex if and only if there exists two vertices x and y such that v is on every $x - y$ path in G.

   ii. Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then prove that $\tau(G) = \tau(G - e) + \tau(G.e)$.

   iii. Explain and write Huffman's algorithm for prefix code.

   iv. Let T be any tree on $k + 1$ vertices. If $\delta(G) \geq k$, then show that G contains a tree isomorphic to T.

4. (a) Attempt any **ONE** question from the following: (8)

   i. Show that a nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

   ii. If u and v are non-adjacent vertices in a graph G such that $deg(u) + deg(v) \geq p$. Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

(b) Attempt any **TWO** questions from the following: (12)

i. Define closure of a graph $C(G)$ and show that it is well defined.

ii. If G is a (p, q) graph with $p \geq 3$ and $q \geq \frac{1}{2}(p-1)(p-2)+2$, then prove that G is Hamiltonian.

iii. Prove that the cube graph Q_k is bipartite $k-$regular graph with 2^k vertices.

iv. Let G_1 and G_2 be two Eulerian graphs with no vertex in common. Let G be a graph obtained by joining some vertex of G_1 to some vertex in G_2. Is G Eulerian? Explain.

5. Attempt any **FOUR** questions from the following: (20)

(a) Define isomorphism of graphs. Give examples of non isomorphic graphs that has
(i) same degree sequence.
(ii) equal number of vertices and equal number of edges.

(b) Show that in a party of 6 or more people, either there are 3 persons who know one another or there are three persons who do not know one another.

(c) Describe the trees produced by Breath First Search (BFS) and Depth First Search (DFS) algorithm for the complete graph K_n where n is positive integer Justify your answer.

(d) Prove that if G is a connected graph of order $p \geq 3$ and G has a cut edge then G contains a cut vertex. Is the converse true? Justify.

(e) Prove that if G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.

(f) Draw Q_n for $1 \leq n \leq 4$ and write the Hamiltonian cycles in them.

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