

Duration 3 Hrs

Marks: 100

- N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. Choose correct alternative in each of the following:

(20)

- i. Which of the following maps $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a metric on \mathbb{R}^2 , where d is defined as, for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$
 - (a) $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$
 - (b) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
 - (c) $d(x, y) = |x_2 - y_2|$
 - (d) $d(x, y) = (x_1 - y_1)^2$
- ii. Consider the discrete metric d defined on a non-empty subset X containing at least two elements. Then, for $p \in X$, which of the following is false?
 - (a) $B(p, 2) = B(p, 3)$
 - (b) $B(p, 1) \subset B(p, 2)$
 - (c) $B(p, 0.5) = B(p, 1)$
 - (d) $B(p, 1) = B(p, 2)$
- iii. Which of the following sets is bounded in \mathbb{R} ?
 - (a) $A = \mathbb{R}$ with respect to the usual metric
 - (b) $A = \mathbb{R}$ with respect to the discrete metric
 - (c) $A = [a, \infty)$ with respect to the usual metric
 - (d) None of the above
- iv. Let $A = \mathbb{Q}, B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $C = \bigcap_{n \in \mathbb{N}} (\frac{-1}{n}, \frac{1}{n})$ be subsets of \mathbb{R} (distance being usual), then,
 - (a) $A^\circ = B^\circ = C^\circ = \emptyset$
 - (b) $\bar{A} = \mathbb{R}; \bar{B} = B; C^\circ = \emptyset$
 - (c) $A^\circ = \emptyset; B^\circ = B; C^\circ = \emptyset$
 - (d) $\bar{A} = \mathbb{R}; B^\circ = \emptyset; C^\circ = \{0\}$
- v. $f : [0, 1] \rightarrow [0, 1]$ is defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 - x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$, then
 - (a) f is continuous on $[0, 1]$ and does not satisfy intermediate value property.
 - (b) f satisfies intermediate value property but f is not continuous.
 - (c) f is continuous only at $x = 1/2$ and $f([0, 1]) = [0, 1]$.
 - (d) none of the above.
- vi. Every Cauchy Sequence is eventually a constant in
 - (a) (\mathbb{N}, d) , where d is the usual distance.
 - (b) (\mathbb{Q}, d) , where d is the usual distance.
 - (c) $(\mathbb{R} \setminus \mathbb{Q}, d)$, where d is the usual distance.
 - (d) None of the above.
- vii. Let (X, d) be a complete metric space, A and B are complete subspaces of (X, d) and $A \cap B$ is nonempty then
 - (a) $A \cup B$ is complete and $A \cap B$ is not.
 - (b) $A \cap B$ is complete and $A \cup B$ is not.
 - (c) $A \cup B$ and $A \cap B$ are complete.
 - (d) none of the above.
- viii. In (\mathbb{R}, d) , where d is usual metric,
 - (a) $\{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{-1, 0, 1\}$ is compact.
 - (b) $\{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$ is not compact.
 - (c) $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$ is compact.
 - (d) None of these.
- ix. In the metric space (\mathbb{Z}, d) , (\mathbb{Z} is the set of integers, d is usual distance), $K \subset \mathbb{Z}$ is compact
 - (a) if and only if K is closed.
 - (b) if and only if K is bounded.
 - (c) if and only if K has a limit point.
 - (d) if and only if $0 \in K$.

- x. Let (x_n) be a sequence in $[0, 1]$ with usual metric from \mathbb{R} . Then, which of the following is **not true**?
- (x_n) has a convergent subsequence.
 - (x_n) is bounded but may not be convergent.
 - (x_n) is Cauchy.
 - (x_n) may have subsequences converging to different limits.

2. (a) Attempt any One of the following: (8)

- Define an open ball $B(x, r)$ in a metric space (X, d) and show that every open ball is an open set. Also give an example to show that the converse need not be true.
- Let (X, d) be a metric space and $A, B \subseteq X$. Show that
 - $A \subseteq B \implies A^\circ \subseteq B^\circ$
 - $(A \cap B)^\circ = A^\circ \cap B^\circ$
 - $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

(b) Attempt any Two of the following: (12)

- State and prove Hausdorff property in a metric space (X, d) .
- Let (X, d) be a metric space and $A \subseteq X$. Show that \bar{A} is a closed set and it is the smallest closed set containing A .
- Prove that (\mathbb{Z}, d) and (\mathbb{Z}, d_1) where d is the usual distance (induced from \mathbb{R}) and d_1 is the discrete metric in \mathbb{Z} , are equivalent metric spaces.
- Answer the following:
 - Consider the subspace $Y = [0, \infty)$ of \mathbb{R} where distance d in \mathbb{R} is usual. Find $B_Y(0, 5)$ an open ball in the subspace (Y, d) .
 - In $(\mathbb{R}, \text{usual})$, show that \mathbb{Q} is neither a closed set nor an open set.

3. (a) Attempt any One of the following: (8)

- State and prove Nested interval theorem in \mathbb{R} .
- Let (X, d) be a metric space and A be a subset of X . Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p .

(b) Attempt any Two of the following: (12)

- Let (X, d) be a metric space and (x_n) be a Cauchy sequence in X . If (x_n) has a convergent subsequence then prove that sequence (x_n) itself is convergent.
- Let (X, d) be a metric space and Y be a non-empty subset of X . Prove that a subset G of Y is open in the subspace (Y, d) if and only if $G = V \cap Y$ where V is an open set in (X, d) .
- Prove that the metric space (\mathbb{R}^2, d_1) is complete where the metric d_1 is given by $d_1((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.
- If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous such that f takes only rational values then show that f is a constant function.

4. (a) Attempt any One of the following: (8)
- Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset need not be compact.
 - Consider the metric space (\mathbb{R}, d) where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that if A is closed and bounded then A satisfies Hein-Borel property.
- (b) Attempt any Two of the following: (12)
- Let A, B be compact subsets of a metric space (X, d) . Show that $A \cup B$ and $A \cap B$ are compact subsets of (X, d) .
 - Prove that a closed subset of a compact metric space is compact.
 - Consider the metric space (\mathbb{R}, d) , where d is the usual distance. Show that $\{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. Is $(0, 1)$ compact? Justify your answer.
 - Prove or disprove:
 - Interior of a compact set is compact.
 - A closed ball $B[x, r]$ in a metric space is compact.

5. Attempt any Four of the following: (20)

- (a) $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms on \mathbb{R}^2 where for $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\|_1 = \sum_{i=1}^2 |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^2 x_i^2}$. Show that $\|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{2} \|x\|_2$ for $x \in \mathbb{R}^2$.
- (b) Define distance of a point p from set A in a metric space (X, d) . If $A \subseteq X$ then show that $|d(x, A) - d(y, A)| \leq d(x, y), \forall x, y \in X$.
- (c) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X . If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2) .
- (d) Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.
- (e) Consider the set $K = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ in a metric space (\mathbb{Q}, d) where d is a usual metric from \mathbb{R} . Is the set K compact in (\mathbb{Q}, d) ? Justify your answer.
- (f) Prove that a subset of a discrete metric space is compact if and only if it is finite.

XX