Duration 2 $\frac{1}{2}$Hrs — Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) Let (X, d) be a metric space and $A \subseteq X$. Show that
   (I) A° is an open set and is the largest open set contained in A .
   (II) A is open if and only if $A = A^\circ$.
   (ii) Define an open ball $B(x, r)$ in a metric space (X, d) and show that every open ball is an open set. Also give an example to show that converse need not be true.

   (b) Attempt any Two of the following: (12)
   (i) Let $(X, \| \|)$ be a normed linear space and $A \neq \emptyset, A \subseteq X$. Show that if $U \neq \emptyset, U \subseteq X$ is an open set then $U + A$ is open.
   (ii) Define distance of a point p from set A in a metric space (X, d). If $A \subseteq X$ then show that $|d(x, A) - d(y, A)| \leq d(x, y), \forall x, y \in X$.
   (iii) Prove or disprove: Every open ball in $(\mathbb{N}, d_1)$ is an open ball in $(\mathbb{N}, d)$ where d_1 is the discrete metric on $\mathbb{N}$ and d is the usual metric.
   (iv) Let d_1, d_2 be metrics on X. Define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Show that d is a metric on X.

2. (a) Attempt any One of the following: (8)
   (i) Show that for a subset F of a metric space (X, d), the following statements are equivalent:
   (I) F is closed
   (II) F contains all its limit points.
   (ii) Let (X, d) be a metric space and Y be a non-empty subset of X. Prove that a subset G of Y is open in the subspace (Y, d) if and only if $G = V \cap Y$ where V is an open set in (X, d) .

   (b) Attempt any Two of the following: (12)
   (i) Let (X, d) be a metric space and $A \subseteq X$. If $G \subseteq X$ is an open set such that $G \cap A = \emptyset$ then show that $G \cap \overline{A} = \emptyset$.
   (ii) Let d_1 and d_2 be metrics on a non-empty set X such that there exists $k_1, k_2 > 0$ such that $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y) \quad \forall x, y \in X$ then show that (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2)
   (iii) Let A be a subset of a metric space (X, d) . Prove that
   (I) $\overline{(X \setminus A)} = X \setminus A^\circ$
   (II) $(X \setminus A)^\circ = X \setminus (\overline{A})$

(iv) Let d and d_1 be equivalent metrics on X. If $(x_n) \longrightarrow p$ in (X, d) then prove that $(x_n) \longrightarrow p$ in (X, d_1).

3. (a) Attempt any One of the following: (8)

(i) Show that if a subset K of $\mathbb{R}^n$ is sequentially compact then it is closed and bounded. (distance being Euclidean)

(ii) If $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ then prove that I is compact (distance in $\mathbb{R}^n$ being Euclidean).

(b) Attempt any Two of the following: (12)

(i) If A, B are compact subsets of $\mathbb{R}^2$ then show that $A + B$ is also a compact subset of $\mathbb{R}^2$.(distance being Euclidean)

(ii) Show that closed subset of compact metric space is compact.

(iii) Show that $(C[0,1], \| \ \|_\infty)$ where $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]$ is not compact by considering the open cover $\{B(0,n) : n \in \mathbb{N}\}$ of $C[0,1]$.

(iv) Prove or disprove:

(I) Interior of a compact set is compact.

(II) Closure of a compact set is compact.

4. Attempt any Three of the following: (15)

(a) State and prove Hausdorff property in a metric space (X, d) .

(b) Show that d is a metric on $\mathbb{N}$, for $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 0 & \text{if} \quad m = n \\ 1 + \dfrac{1}{m+n} & \text{if} \quad m \neq n \end{cases}$$

(c) Which of the following are dense subsets of $\mathbb{R}$ with usual distance? Justify your answer.

(I) $\mathbb{Z}$ (II) $\mathbb{R}\backslash\mathbb{Q}$ (III) $\mathbb{R}\backslash\mathbb{Z}$

(d) Let (X, d) be a metric sapce (X, d). Show that every Convergent sequence X is Cauchy.

(e) Give an example of two metrics d_1 and d_2 on X such that (X, d_1) is compact but (X, d_2) is not compact.

(f) Which of the following subsets of $(\mathbb{R}^2, d)$, (d being Euclidean distance) are compact? Justify your answer.

(i) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(ii) $B = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$