2 $\frac{1}{2}$ Hours] **[Total Marks: 75**

N.B.: (1) **All questions are compulsory**.
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry such that $T(0) = 0$ then, show that T is an orthogonal linear transformation. (8)
   ii. state and prove the Cayley Hamilton theorem. (8)

   (b) Answer any **TWO**
   i. Let V be a finite dimensional real vector space and W be a subspace of V. Show that $dim\, V/W = dim\, V - dim\, W$. (6)
   ii. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x) = AX$(X being a column vector in $\mathbb{R}^3$). Find $kerT$,a basis of $kerT$ and $\mathbb{R}^3/kerT$. Also find ImT. (6)
   iii. Show that similar matrices have same characteristic polynomial. Is the converse true? Justify. (6)
   iv. If $A_{2\times 2}$ is a nilpotent real matrix then prove that $A^m = 0$ for all positive integers $m > 1$. (6)

2. (a) Answer any **ONE**
   i. Show that a real matrx with n eigen values is similar to an upper triangular matrix of order n with the eigen values on diagonal. (8)
   ii. Show that minimal polynomial of a real matrix $A_{n\times n}$ divides every polynomial which annihilates A. Further prove that α is a root of the minimal polynomial of matrix A if and only if α is a characteristic root of A. (8)

   (b) Answer any **TWO**
   i. Show that eigen vectors $v_1, v_2, \cdots, v_k$ corresponding to distinct eigen values $\lambda_1, \lambda_2, \cdots, \lambda_k$ respectively of a square matrix A are linearly independent. (6)
   ii. If A, B are $n \times n$ real matrices and A is non-singular then show that AB and BA have same eigen values. (6)
   iii. Let $A_{3\times 3}$ be a real matrix and 1, -1, 3 be its eigen values. Which of A, $A^2 - A$, $A^2 - I$, $A^2 - 2A$, $A^2 + 3A$, $A^2 - 3A$ matrices are nonsingular? Justify your answer. (6)

[P.T.O.]

iv. Let $A_{n\times n}$ be a real matrix. if A has n distinct characteristic roots, then prove that the characteristic polynomial of $A =$ the minimal polynomial of A. (6)

3. (a) Answer any **ONE**

i. Show that an $n \times n$ matrix A is diagonalizable if and only if A has n eigen values where algebraic multiplicity of each eigen value coincides with its geometric multiplicity. (8)

ii. Show that a quadratic form Q is positive definite if and only if all eigen values of associated symmetric matrix are positive. (8)

(b) Answer any **TWO**

i. Show that a real matrix $A_{n\times n}$ with distinct eigen values is diagonalizable. (6)

ii. Let A be a real symmetric matrix. Show that $\langle AX, Y\rangle = \langle X, AY\rangle$ for all $X, Y \in \mathbb{R}^n$. Hence or otherwise, prove that eigen vectors corresponding to distinct eigen values of a real symmetric matrix are mutually orthogonal. (6)

iii. If A is a $n \times n$ diagonalizable matrix with eigen values 1 and -1, show that $A = A^{-1}$. (6)

iv. Show that a non-zero $n\times n$ matrix of rank 1 is diagonalisable with eigen values 0, tr A. (6)

4. Answer any **THREE**

(a) Find the orthogonal transformations in $\mathbb{R}^3$ which represent reflection with respect to $x + y - z = 0$. (5)

(b) Show that $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x, y, z) = (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - 2, \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + 3, z + 2)$ is an isometries. Express it as a composite of an orthogonal transformation and a translation. . (5)

(c) Let V be a vector space of finite dimension and $T : V \to V$ be a linear transformation. Show that eigen space corresponding to any eigen value of T is invariant under T. (5)

(d) Let $A_{n\times n}$ be a matrix with all the entries as 1. Find eigen values and the corresponding eigen spaces of A. (5)

(e) Prove that if every non-zero vector of $\mathbb{R}^n$ is an eigen vector of $A_{n\times n}$ then A is a $n \times n$ scalar matrix. (5)

(f) By applying rotation of coordinate axes reduce the conic $2x^2 - 4xy - y^2 + 8 = 0$ into standard form. Hence identify it. (5)

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