3 Hours] **[Total Marks: 100**

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. Fill in the blank by choosing the correct option.

   i. Let $V = \mathbb{R}^3, W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$, then dim $V/W =$ ———. (2)
   (a) 2 (b) 3 (c) 1 (d) None of these.

   ii. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is ——— (2)
   (a) an orthogonal matrix of reflection
   (b) an orthogonal matrix of rotation
   (c) not an orthogonal matrix
   (d) None of these.

   iii. Let $A = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$, then $A^{-1} =$ ———. (2)
   (a) $A - 6I$ (b) $A + 6I$ (c) $A - I$ (d) None of these.

   iv. If characteristic polynomial of A is $t^2 + a_1 t + a_0$ and characteristic polynomial of A^{-1} is $t^2 + b_1 t + b_0$.Then $a_0 b_0 =$——— (2)
   (a) 0 (b) 1 (c) -1 (d) None of these.

   v. If 2 is an eigen value of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ then one of the eigen values of $T^2 - 3T$ is ———- (2)
   (a) 1 (b) 2 (c) -2 (d) None of these.

   vi. Let $A_{3\times3}$ be a real matrix of rank 1, then the eigen values of A are ———- (2)
   (a) 0 and 1 (b) 0 and tr A
   (c) 0 and det A (d) None of these.

   vii. The minimal polynomial of the diagonal matrix $A =$ diag $\{1, -1, 1, -1\}$ is——— (2)
   (a) $x^2 + 1$ (b) $x^2 - 1$ (c) $(x^2 - 1)^2$ (d) None of these.

   viii. If non-zero, non-diagonal $A, B \in M_2(\mathbb{R})$ such that $A^2 = I$, $B^2 = 0$, then ———- (2)
   (a) only A is diagonalisable. (b) only B is diagonalisable.
   (c) both A and B are diagonalisable. (d) None of these.

[Turn over]

ix. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is ——— (2)

(a) diagonalisable but not orthogonally diagonalisable.
(b) orthogonally diagonalisable
(c) not diagonalisable
(d) None of these.

x. Rank and signature of the quadratic from $Q(x) = 2x_1^2 + 2x_2^2 - 2x_1x_2$ are ———- (2)

(a) 2 and 2 (b) 2 and 0
(c) 2 and -2 (d) None of these.

2. (a) Attempt any **ONE**

i. Let V be a finite dimensional inner product vector space and $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent. (8)

(p) T is orthogonal.
(q) $\|T(X)\| = \|X\|$ for all $X \in V$.
(r) If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V, then $\{T(e_i)\}_{i=1}^n$ is also an orthonormal basis of V.

ii. State and prove the 'First Isomorphism Theorem of vector space' (Fundamental theorem of vector space homomorphism). (8)

(b) Attempt any **TWO**

i. Show that any orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation about origin or a reflection about a line passing through origin. (6)

ii. Let $A_{n\times n}$ be a real matrix. If a_0 is the constant term of the polynomial det $(xI_n - A)$ then show that $a_o = (-1)^n$ det A. (6)

iii. Let $A_{7\times 7}$ be a diagonal matrix over $\mathbb{R}$ with characteristic polynomial $(t+4)^3(t-3)^4$. Let $W = \{B \in M_7(\mathbb{R}) : AB = BA\}$. Find dim $M_7(\mathbb{R})/W$. (6)

iv. Let A be $n \times n$ real matrix. Express the characteristic polynomial of $aI + A$ in terms of the characteristic polynomial of A where $a \in \mathbb{R}$. Hence or otherwise show that, if $A_{n\times n}$ is nilpotent then the characteristic polynomial of $A - I_n$ is $(x-1)^n$. (6)

3. (a) Attempt any **ONE**
   i. Let $A_{n\times n}$ be a real matrix and $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of A with $X_1, X_2, \cdots, X_k$ as corresponding eigen vectors, then show that $X_1, X_2, \cdots, X_k$ are linearly independent. (8)
   ii. Define minimal polynomial of a suare matrix. Show that α is a root of the minimal polynomial of matrix A if and only if α is a characteristic root of A. (8)

   (b) Attempt any **TWO**
   i. Define invariant subspace. Let V be a finite dimension vector space and $T : V \to V$ be a linear transformation. Show that $ker\, T, Im\, T$ are invariant under T. (6)
   ii. Let A and B be $n \times n$ real matrices. Prove that if A and B are similar then characteristic polynomial of A = characteristic polynomial of B. Is the converse true? Justify. (6)
   iii. Let λ_0 be an eigen value of $n \times n$ matrix A. Show that any non-zero column of adj $(A - \lambda_0 I)$ is an eigen vector of A corresponding to λ_0. (6)
   iv. Find the characteristic polynomial and the minimal polynomial of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$. (6)

4. (a) Attempt any **ONE**
   i. Show that an $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigen vectors of A. (8)
   ii. Show that any real symmetric matrix is orthogonally diagonalizable. (8)

   (b) Attempt any **TWO**
   i. Show that every quadratic form $Q(x_1, x_2, \cdots, x_n)$ over $\mathbb{R}$ can be reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of the variables $X = PY$, $X = (x_1, x_2 \cdots, x_n)^t, y = (y_1, y_2, \cdots, y_n)^t$ and P is an $n \times n$ orthogonal matrix. (6)
   ii. Show that eigen vectors associated to distinct eigen values of a real symmetric matrix are orthogonal. (6)
   iii. Show that $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is diagonalizable if and only if $b = 0$ or $a \neq d$. (6)

[Turn over]

iv. Let A be a square matrix of order n such that $A^2 = A$. Show that A is diagonalizable. (6)

5. Attempt any **THREE**

(a) If $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find $A^4 - 4A^3 - A^2 - 4A - 20I$ using the Cayley Hamilton theorem. (5)

(b) Let A and B be $n \times n$ real matrices. If A and AB are orthogonal matrices then prove that B and BA are both orthogonal matrices. (5)

(c) Find the eigenvalues and the bases of the corresponding eigen spaces for $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. (5)

(d) If $(x-1)(x+2)^2$ is the characteristic polynomial of a matrix $A_{3\times 3}$ then find the characteristic polynomial of (i) A^{-1} (ii) A^t (iii) A^2. (5)

(e) Show that $A = \begin{bmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{bmatrix}$ is diagonalisable. (5)

(f) Identify the conic $2x^2 - 4xy - y^2 + 8$. (5)

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