2 $\frac{1}{2}$ Hours] **[Total Marks: 75**

N.B.: (1) **All questions are compulsory**.

(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**

i. Let V be a finite dimensional inner product space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent. (8)

p. T is orthogonal.

q. $\|T(X)\| = \|X\|$ for all $X \in V$.

ii. State and prove the first isomorphism theorem for vector space. (8)

(b) Answer any **TWO**

i. Let V be a vector space of finite dimension and W be a subspace of V. Prove that dim V/W = dim $V-$ dim W. (6)

ii. Let W be an n dimensional inner product space and let W be a $n-1$ dimensional subspace of V. Let u be a unit vector orthogonal to W. Show that $T : V \to V$ defined by $T(x) = x - 2\langle x, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w, \quad \forall w \in W$. (6)

iii. State the Cayley-Hamilton theorem. Using the theorem find $A^5 - 2A^4 - A^3 + 2A^2 + A - I$, where $A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$. (6)

iv. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\alpha((x, y)) = (ax + by + e, cx + dy + f)$, where $a, b, c, d, e, f \in \mathbb{R}$. Show that α is an isometry if and only if $a^2 + c^2 = 1, b^2 + d^2 = 1, ab + cd = 0$. (6)

2. (a) Answer any **ONE**

i. Show that every $n \times n$ real matrix with n eigen values is similar to an upper triangular matrix. (8)

ii. Let A be $n \times n$ real symmetric matrix. Show that the following statements are equivalent. (8)

(p) $\langle AX, X\rangle > 0$ for all non zero $X \in \mathbb{R}^n$.

(q) Each eigen values of A is positive.

(b) Answer any **TWO**

i. Define orthogonally diagonalizable matrix. Show that $A_{n\times n}$ is orthogonally diagonalizable if and only is $\mathbb{R}^n$ has an orthonormal basis of eigen vectors of A. (6)

[P.T.O.]

ii. Show that a quadratic form $Q[X]$ can be reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by orthogonal change of variable $X = PY$, $X = [x_1\ x_2\ \cdots x_n]^t$, $Y = [y_1\ y_2\ \cdots y_n]^t$ and orthogonal matrix $P_{n\times n}$. (6)

iii. Show that characteristic roots of real symmetric matrix are real. (6)

iv. Let $A_{n\times n}$ be a non-zero real matrix such that $A^k = 0$ for some $k \in \mathbb{N}$. Show that characteristic polynomial of A is λ^n. (6)

3. (a) Answer any **ONE**

i. Define a cyclic group. Show that subgroup of a cyclic group is cyclic. Give an example to show that the converse is not true. (8)

ii. State and prove the Lagrange's theorem. (8)

(b) Answer any **TWO**

i. Show that an infinite cyclic group has only two generators. (6)

ii. Prove that if H and K are subgroups of a group G then HK is a subgroup of group G if and only if $HK = KH$. (6)

iii. Let $G = \mathbb{Z}_{20}$. List all the subgroups of G and also list the generators of each subgroup. (6)

iv. Show that the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q} - \{0\}, \cdot)$ are not isomorphic. (6)

4. Answer any **THREE**

(a) Let V be a vector space and W be a subspace of V. Show that $W + x = W + y$ for $x, y \in V$ if and only if $y - x \in V$ (5)

(b) If A is a 3×3 orthogonal matrix such that $det(A) = 1$. Show that 1 is an eigen value of A. (5)

(c) Let A be a diagonalizable matrix. Show that $f(A)$ is also diagonalizable where $f(x)$ is a polynomial over $\mathbb{R}$. (5)

(d) Show that $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is diagonalizable but $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is not diagonalizable. (5)

(e) Show that $(\mathbb{Q}^+, \circ)$ is a group where $a \circ b = ab/7$, $\forall\, a, b \in \mathbb{Q}^+$. (5)

(f) Prove that $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ defined by $f(A) = \det A$ is a group homomorphism. Show that f is onto but not one-one. (5)

****