Duration:[3 Hours] **[Total Marks: 100]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. The total number of edges in a complete graph on p vertices is
(a) p (b) $C(p, 2)$ (c) p^2 (d) $p(p-1)$

ii. If a graph G has 21 edges, 3 vertices of degree 4 and other vertices of degree 3 then the number of vertices in graph G is
(a) 10 (b) 13 (c) 14 (d) 9

iii. Which of the following is a correct statement?
(a) Isomorphic graphs have same degree sequence;
(b) Non isomorphic graphs have different degree sequences;
(c) Two graphs are isomorphic if and only if they have same degree sequences;
(d) Two isomorphic graphs need not have same degree sequence.

iv. Value of x, if the degree sequence 8, x, 7, 6, 6, 5, 4, 3, 3, 1, 1, 1 is graphical, is
(a) 8 or 7 (b) 8 (c) 7 (d) None of these

v. The number of cut vertices of a tree on $n \geq 3$ vertices which is a path is
(a) $n-2$ (b) n (c) 2 (d) $n-1$

vi. A spanning tree T of a graph G has
(a) $V(T) = V(G)$ and $E(T) \subseteq E(G)$
(b) $V(T)) \subseteq V(G)$ and $E(T) \subseteq E(G)$
(c) $V(G)) \subseteq V(T)$ and $E(G) \subseteq E(T)$
(d) $V(T) = V(G)$ and $E(G) \subseteq E(T)$

vii. $K_{5,5}$ is
(a) Hamiltonian but not Eulerian (b) Eulerian but not Hamiltonian
(c) Both Hamiltonian and Eulerian (d) Neither Hamiltonian nor Hamiltonian

viii. Of the following graphs, the non - Hamiltonian graph is
(a) Cycle on n vertices (b) K_n, $n \geq 3$
(c) Tree on n vertices (d) Cube graph Q_k, $k > 1$

ix. Q_n is a graph with
(a) 2^n vertices (b) $n2^{n-1}$ edges
(c) Bipartite and Connected (d) All of the above.

x. If G is a graph on p vertices and $Cl(G)$ is closure of graph G then closure of G, $Cl(G)$ is
(a) Super graph of G (b) subgraph of G
(c) compliment of G (d) None of these.

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2. (a) Attempt any **ONE** question from the following: (8)
   i. Show that a nontrivial graph G is bipartite if and only if it contains no odd cycle.
   ii. State and prove $Havel - Hakimi$ theorem for degree sequence of a graph G.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Define Adjacency matrix A of a graph G. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and adjacency matrix $A = [a_{ij}]$. Show that the entry $a_{ij}^{(k)}$ in i^{th} row and j^{th} column of A^k is the number of distinct $v_i - v_j$ walks of length k in G.
   ii. Let G be a simple graph and $\delta(G) \geq 2$, then show that there exists a cycle of length at least $\delta(G) + 1$ in G.
   iii. If G is simple graph on p vertices and $\delta(G) \geq \frac{p-1}{2}$, then prove that G is a connected graph where $\delta(G)$ denotes the minimum degree of G.
   iv. Show that Dijkstra's algorithm produces the shortest path.

3. (a) Attempt any **ONE** question from the following: (8)
   i. State and prove Cayley's formula for spanning trees.
   ii. Define a spanning tree of a graph G. Show that a graph G is connected if and only if it has a spanning tree.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Define a cut vertex of a graph G. Show that every nontrivial graph contains at least two vertices which are non-cut vertices.
   ii. Describe the trees produced by Breath First Search (BFS) and Depth First Search (DFS) algorithm for the complete graph K_n where n is positive integer. Justify your answer.
   iii. Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then prove that $\tau(G) = \tau(G - e) + \tau(G.e)$.
   iv. Use Huffman coding to encode these symbols with the given frequencies:
   a : 0.20, b : 0.10, c : 0.15, d : 0.25, e : 0.30. What is average number of bits required to encode a character?

4. (a) Attempt any **ONE** question from the following: (8)
   i. If u and v are non-adjacent vertices in a graph G such that $deg(u) + deg(v) \geq p$. Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.
   ii. Prove that a connected graph G contains Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, prove that each Eulerian trail of G begins at one of these odd vertices and ends at the other.

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(b) Attempt any **TWO** questions from the following: (12)

i. Define a cube graph Q_k. Show that the cube graph Q_k, $k \geq 2$ is a Hamiltonian graph.

ii. If G is a graph on p vertices with $p \geq 3$ such that $deg(u) + deg(v) \geq p - 1$ for every pair of non adjacent vertices u and v in G, then show that G contains a Hamiltonian path.

iii. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the $27 \times 1 \times 1$ sub-cubes. If he starts at one corner and always move on to an uneaten sub-cube, can he finish at the centre of the cube?

iv. If G is Hamiltonian graph then for every nonempty proper subset S of $V(G)$, prove that $\omega(G - S) \leq |S|$. Give an example of a graph which satisfies the above condition but not Hamiltonian.

5. Attempt any **FOUR** questions from the following: (20)

(a) If G is simple graph with p vertices, q edges and k components, then prove that $q \geq p - k$.

(b) If $G(p, q)$ is a graph with p vertices and q edges, then prove that $\sum_{v \in v(G)} degv = 2q$. Hence prove that every graph has an even number of odd vertices.

(c) Describe Kruskal's algorithm for finding minimum spanning tree in a connected weighted graph.

(d) Prove that if G is a connected graph of order $p \geq 3$ and G has a cut edge then G contains a cut vertex. Is the converse true? Justify.

(e) Define closure of a graph $C(G)$ and show that a simple graph is Hamiltonian if and only if its closure is Hamiltonian.

(f) Define a line graph of a graph G. Show that the line graph a simple graph G is a path if and only if G is a path.

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