Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question from the following: (8)
   i. Define a self complementary graph. If G is self complementary graph of order p, show that G is connected and $p \equiv 0$ or $1 (\mod 4)$
   ii. Show that a nontrivial graph is bipartite if and only if it contains no odd cycle.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Define adjacency matrix of a graph G. If $(A^n) = (a_{ij}^n)$ is the n^{th} power of adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, \ldots v_n\}$, then show that the number of triangles in G is $\frac{1}{6}$trace of A^3.
   ii. State $Havel - Hakimi$ theorem for degree sequence of a graph. Check whether the sequence $6, 6, 5, 4, 3, 3, 1$ is graphical or not? If graphical, construct a graph, for which the given sequence is a degree sequence of the graph. If not, Justify you answer.
   iii. Show that a connected (p, q) graph G contains a cycle if and only if $q \geq p$.
   iv. If G is graph of order n with $\delta(G) \geq (n-1)/2$, then show that G is connected. Is the bound $(n-1)/2$ sharp?, that is, in this case, can $(n-1)/2$ be replaced by$(n-2)/2$?

2. (a) Attempt any **ONE** question from the following: (8)
   i. Define a spanning tree of a graph G. Show that a graph is connected if and only if it has a spanning tree.
   ii. Let G be a (p, q) graph. Prove that following statements are equivalent.
   1) G is tree.
   2) G is acyclic and $p = q + 1$.
   3) G is connected and $p = q + 1$.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Show that each label spanning tree with n vertices corresponds to a unique vector $s = (s_1, s_2, \ldots .s_{n-2})$ where $s_i \in \{1, 2, \ldots, n\}$ for $i = 1, 2, \ldots, n$
   ii. Let T be any tree on $k + 1$ vertices. If $\delta(G) \geq k$, then show that G contains a tree isomorphic to T.
   iii. Define a cut vertex of a graph G. Show that vertex v is a cut vertex if and only if there exists two vertices x and y such that v is on every $x - y$ path in G.
   iv. Describe the trees produced by Breath First Search (BFS) and Depth First Search (DFS) algorithm for the complete graph K_n where n is positive integer. Justify your answer.

3. (a) Attempt any **ONE** question from the following: (8)

   i. If G is graph on p vertices with $p \geq 3$ and $\delta(G) \geq \frac{p}{2}$ where $\delta(G)$ denotes the minimum degree of G, then show that G contains a Hamiltonian cycle.

   ii. Prove that a connected graph G contains Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, prove that each Eulerian trail of G begins at one of these odd vertices and ends at the other.

   (b) Attempt any **TWO** questions from the following: (12)

   i. Define a line graph of a graph. Show that if the simple graphs G_1 and G_2 are isomorphic, then its line graphs $L(G_1)$ and $L(G_2)$ are isomorphic. Is converse true? Justify.

   ii. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the $27 \times 1 \times 1$ sub-cubes. If he starts at one corner and always move on to an uneaten sub-cube, can he finish at the centre of the cube?

   iii. Define closure $C(G)$ of a graph G. Show that a simple graph G is Hamiltonian if and only if its closure is Hamiltonian.

   iv. If G is a graph on p vertices with $p \geq 3$ such that $deg(u) + deg(v) \geq p - 1$ for every pair of non adjacent vertices u and v in G, then show that G contains a Hamiltonian path.

4. Attempt any **THREE** questions from the following: (15)

   (a) Define complement of a graph G. For any graph G with at least 6 vertices, prove that either G or G^c contains a triangle.

   (b) Show that every $u - v$ walk W contains $u - v$ path.

   (c) Explain and write Huffman's algorithm for prefix code.

   (d) Prove that a connected graph G is a tree if and only if every edge of G is a cut edge.

   (e) If G is Hamiltonian graph then for every nonempty proper subset S of $V(G)$, prove that $\omega(G - S) \leq |S|$. Give an example of a graph which satisfies the above condition but not Hamiltonian.

   (f) Show that the cube graph Q_k, $k \geq 2$ is a Hamiltonian graph.

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