Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indcate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. Show that every planar graph is 5-vertex colorable.
   ii. State and prove Euler's fromula for planar graphs. Hence or otherwise prove that the minimum degree of a simple planar graph is ≤ 5.

   (b) Attempt any **TWO** questions: (12)
   i. Define vertex chromatic number $\chi(G)$ of a simple finite graph G. Prove that $\chi(G) \leq \Delta(G) + 1$ where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G.
   ii. If G is a planar (p, q) graph in which every face is bounded by a cycle of length at least n, then show that $q \leq \frac{n(p-2)}{n-2}$.
   iii. For a simple graph G of order p and size q, prove that $\pi_k(G)$, the chromatic polynomial of the graph G, is monic polynomial of degree p in k with integer coefficients and constant term zero.
   iv. Show that for a cycle C_n of length n, the chromatic polynomial π_k is given by $\pi_k(C_n) = (k-1)^n + (-1)^n(k-1)$ for $n \geq 3$.

2. (a) Attempt any **ONE** question: (8)
   i. State and prove Max Flow-Min Cut Theorem.
   ii. State and prove the necessary and sufficient condition for a family of n sets to have System of Distinct Representative.

   (b) Attempt any **TWO** questions: (12)
   i. Define $val(f)$, value of flow f. If f is flow in a network N and P is any $f-$incrementing path, then show that there exists a revised flow f' such that $valf' > valf$.
   ii. Show that the number of different system of distinct representatives for the family $A_i = \{1, 2, \ldots, n\} - \{i\}$, $1 \leq i \leq n$ is D_n, the number of derangements on n symbols.
   iii. Prove that for any flow f and any cut $(S, \overline{S})$, $val(f) = f^+(S) - f^-(S)$.
   iv. If $\{A_1, A_2, \ldots, A_n\}$ be a family of set, then prove that the largest number of sets of the family which together have a system of distinct representative equals the minimum value of expression $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + (n-k)$ for all choices of $k = 1, 2, \ldots n$ and all choices of $i_1, i_2, \ldots, i_k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

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3. (a) Attempt any **ONE** question: (8)

   i. Derive the recurrence relation for number of ways of dividing a $(n+1)$ - sided convex polygon into triangular regions by inserting diagonals that do not intersect in the interior and prove using generating function that the solution to this recurrence relation is a Catalan Number.

   ii. Derive the recurrence relation to climb a staircase with n steps by taking 1 or 2 steps at a time and solve it using generating function.

   (b) Attempt any **TWO** questions: (12)

   i. Let B denotes a forbidden chess board in which a special square $*$ has been identified. Let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square $*$ is removed from the board, then prove that $R(x,B) = xR(x,D) + R(x,E)$.

   ii. Derive the recurrence relation for the number of regions into which the plane is divided by n straight lines, no two of which are parallel and no three of which are concurrent. Furthermore using generating function, show that the solution of the above recurrence is $\frac{n(n+1)}{2} + 1$

   iii. Solve recurrence relation $a_n = a_{n-1} + 1$ for all $n \geq 2$ subject to initial conditions $a_1 = 1$ using generating function.

   iv. Show that the number of nonnegative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = r$ is given by $\binom{r+k-1}{r}$.

4. Attempt any **THREE** questions: (15)

   (a) Show that a connected graph G on n vertices is a tree if and only if the chromatic polynomial of G is $k(k-1)^{n-1}$.

   (b) Show that $\chi'(G) \geq \Delta(G)$ where $\chi'(G)$ denotes edge chromatic number and $\Delta(G)$ denotes the maximum degree of G. Give an example of the graph for which $\chi'(G) = \Delta(G)$.

   (c) If f is any flow and K be any cut in a network N with $val(f) = cap(K)$ then show that f is maximum flow and K is minimum cut.

   (d) Define System of distinct representatives for a family of sets. Let $A = (A_1, A_2, A_3, A_4, A_5, A_6)$, where $A_1 = \{1,2,3\}$, $A_2 = \{1,2,3,4,5\}$, $A_3 = \{1,2\}$, $A_4 = \{2,3\}$, $A_5 = \{1\}$, $A_6 = \{1,3,5\}$. Does family A have an System of Distinct Representative? If not, what is the largest number of sets in the family with an System of Distinct Representative?

   (e) Prove that if B is a board of darkened squares that decomposes into the two disjoint sub boards B_1 and B_2 then prove that $R(x,B) = R(x,B_1)R(x,B_2)$, where $R(x,B)$ is a rook polynomial for board B.

   (f) If h_n denote the number of nonnegative integral solutions of the equation $3e_1 + 4e_2 + 2e_3 + 5e_4 = n$, find generating function for h_n.

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