Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indcate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. Prove that a graph $G(p, q)$ with $p \geq 2$ is 2-connected if and only if any two vertices are connected by at least two internally disjoint paths.
   ii. If $\pi_k(G)$ denotes the chromatic polynomial of a (p, q) graph G then prove that
      (a) The coefficient of k^p in $\pi_k(G)$ is 1.
      (b) The constant term of $\pi_k(G)$ is zero.
      (c) The terms of $\pi_k(G)$ are alternate in sign.
      (d) The coefficient of k^{p-1} is $-q$ where q is number of edges of G.

   (e) Attempt any **TWO** questions: (12)
   i. Define vertex chromatic number of a graph G. If G is a (p, q) graph, then prove that $\chi(G) \geq \frac{p^2}{p^2-2q}$ where $\chi(G)$ denotes the vertex chromatic number of G.
   ii. If G is cubic graph, then show that $\kappa(G) = \kappa'(G)$ where $\kappa(G)$ denote the vertex connectivity and $\kappa^{'}(G)$ denotes the edge connectivity of a graph G.
   iii. State vizing theorem for edge coloring of graph. Show that $\chi'(G) \geq \Delta(G)$ where $\chi'(G)$ denotes edge chromatic number and $\Delta(G)$ denotes the maximum degree of G. Give an example of the graph for which $\chi'(G) = \Delta(G)$.
   iv. If G is a cycle on n vertices then show that $\pi_k(G) = (k-1)^n + (-1)^n(k-1)$.

2. (a) Attempt any **ONE** question: (8)
   i. State and prove Max Flow - Min Cut Theorem.
   ii. Show that every planar graph is 5 vertex colorable.

   (b) Attempt any **TWO** questions: (12)
   i. Let f be a flow in a network N and P be any f-incrementing path then show that there exist a revised flow f' such that $val(f') = val(f) + \epsilon(p)$
   ii. State and prove Euler theorem for planar graph.
   iii. Show that edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.
   iv. Show that the complete graph K_5 and complete bipartite graph $K_{3,3}$ are nonplanar.

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3. (a) Attempt any **ONE** question: (8)

   i. State and prove Hall's (Marriage) Theorem for a System of Distinct Representatives.

   ii. Derive the recurrence relation for number of ways of dividing a $n+1-$sided convex polygon into triangular regions by inserting diagonals that do not intersect in the interior and prove using generating function that the solution to this recurrence relation is a Catalan Number.

   (b) Attempt any **TWO** questions: (12)

   i. Show that a matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.

   ii. Let B denotes a forbidden chess board in which a special square $*$ has been identified and let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square $*$ is removed from the board, then prove that $R(x,B) = xR(x,D) + R(x,E)$.

   iii. Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \dots)^5$. What is the coefficient of x^r?

   iv. Solve recurrence relation $a_n = 3a_{n-1}$, $(n \geq 1)$ subject to initial condition with $a_0 = 1$ using generating function.

4. Attempt any **THREE** questions: (15)

   (a) Lel $\pi_k(G)$ denote the chromatic polynomial of the graph G. If G is simple graph then prove that $\pi_k(G) = \pi_k(G-e) - \pi_k(G.e)$ where e is an edge of G.

   (b) Define $k-$critical graph. If G is $k-$critical graph then show that $\delta(G) \geq k-1$, where $\delta(G)$ denotes the minimum degree of G.

   (c) Show that there is at least one face of every polyhedron is bounded by an $n-$cycle for some $n = 3, 4$ or 5.

   (d) Prove that for any flow f and any cut $(S, \overline{S})$, $val(f) = f^+(S) - f^-(S)$.

   (e) Define System of distinct representatives for a family of sets and hence determine the number of distinct system of representatives for the family $A_1 = \{1,2\}, A_2 = \{2,3\}, A_3 = \{3,4\}, A_4 = \{4,5\}, A_5 = \{5,1\}$. Generalize your result for n.

   (f) Find the rook polynomial for the following $\{(1,3),(1,4),(2,1),(2,2),(3,3),(4,3),(5,2),(5,5)\}$

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