Q. P. Code: 19409

Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indcate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. Show that a nontrivial graph is bipartite if and only if it contains no odd cycle.
   ii. If $(A^n) = (a^n_{ij})$ is the n^{th} power of adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, \dots v_n\}$, then prove that
   1) a^2_{ij}, $i \neq j$ is the number of $v_i - v_j$ path of length 2.
   2) $a^2_{ii} = deg(v_i)$
   3) $\frac{1}{6}$trace of A^3 is the number of triangles in G.

   (b) Attempt any **TWO** questions: (12)
   i. Define a complement of a graph G. For any graph G with at least 6 vertices, prove that either G or G^c contains a triangle.
   ii. Explain and write Dijkstra's algorithm to find the shortest path in a graph G.
   iii. Show that a graph G is disconnected if and only if its vertex set V can be partitioned into two subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in the subset V_1 and the other in the subset V_2.
   iv. Show that the number of edges of a simple graph with n vertices and k components cannot exceed $\frac{(n-k)(n-k+1)}{2}$.

2. (a) Attempt any **ONE** question: (8)
   i. Define a spanning tree of a graph G. Show that a graph is connected if and only if it has a spanning tree.
   ii. Let G be a (p, q) graph. Prove that following statements are equivalent.
   a) G is tree.
   b) G is acyclic and $q = p - 1$.
   c) G is connected and $q = p - 1$.

   (b) Attempt any **TWO** questions: (12)
   i. Show that each label spanning tree with n vertices corresponds to a unique vector $s = (s_1, s_2, ...s_{n-2})$ where $s_i \in \{1, 2, ..., n\}$ for $i = 1, 2, ..., n$
   ii. Let T be any tree on $k + 1$ vertices. If $\delta(G) \geq k$, then show that G contains a tree isomorphic to T.
   iii. Use Huffman coding to encode these symbols with the given frequencies: a : 0.08, b : 0.10, c : 0.12, d : 0.15, e : 0.20, f : 0.35. What is average number of bits required to encode a character?
   iv. Prove that a connected graph G is a tree if and only if every edge of G is a cut edge.

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3. (a) Attempt any **ONE** question: (8)

   i. Prove that a connected graph G contains Eulerian trail if and only if exactly two vertices of G have odd degree.

   ii. If G is a graph on p vertices with $p \geq 3$ such that $deg(u) + deg(v) \geq p$ for every pair of non adjacent vertices u and v in G, then prove that G is Hamiltonian.

   (b) Attempt any **TWO** questions: (12)

   i. Define a cube graph. Show that the cube graph Q_k, $k \geq 2$ is a Hamiltonian graph.

   ii. Let G be a connected graph with $2n$ odd vertices with $n \geq 1$. Show that $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots E_n$ so that $< E_i >$ is an open trail for each i.

   iii. If G is Hamiltonian graph then for every nonempty proper subset S of $V(G)$, prove that $\omega(G - S) \leq |S|$. Is converse true? Justify.

   iv. If G is a (p, q) graph with $p \geq 3$ and $q \geq \frac{1}{2}(p-1)(p-2) + 2$, then prove that G is Hamiltonian.

4. Attempt any **THREE** questions: (15)

   (a) If G is a graph of order p and size q, then prove that $\sum_{v \in v(G)} degv = 2q$. Hence prove that every graph has an even number of odd vertices.

   (b) If a graph G contains a $u - v$ walk of length l, then show that G contains a $u - v$ path of length at most l.

   (c) Define minimum spanning tree of a graph. Describe Kruskal's algorithm for finding minimum spanning tree in a connected weighted graph.

   (d) Describe the trees produced by Breath First Search (BFS) and Depth First Search (DFS) algorithm for the wheel graph W_n starting at the vertex of degree n where n is integer with $n \geq 3$.

   (e) Let G be a simple graph with $p \geq 3$. If closure of G is complete, show that G is Hamiltonian.

   (f) Show that the line graph a simple graph G is a path if and only if G is a path.

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