**Q. P. Code : 19566**

**2 $\frac{1}{2}$ Hours]** **[Total Marks: 75**

N.B.: (1) **All questions are compulsory**.
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. Let $V$ be a finite dimensional inner product space. If $f : V \to V$ is a function such that (p) $f(0) = 0$ (q) $\|f(X) - f(Y)\| = \|X - Y\|$, $\forall\ X, Y \in V$, then show that $f$ is an orthogonal linear transformation. (8)
   ii. State and prove the Cayley-Hamilton theorem. (8)

   (b) Answer any **TWO**
   i. Let $W$ be a subspace of a finite dimension real vector space $V$. Show that dim $V/W =$ dim $V-$ dim $W$. (6)
   ii. Show that a $2 \times 2$ orthogonal matrix with determinant $= 1$ is a matrix of rotation. (6)
   iii. Find an orthogonal transformation in $\mathbb{R}^3$ which represents reflection with respect to the plane $x - y + z = 0$. (6)
   iv. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $< u, v > = 0 \Rightarrow < T(u), T(v) > = 0\ \forall\ u, v \in \mathbb{R}^2$ then show that $T = \alpha S$, where $\alpha \in \mathbb{R}$ and $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orthogonal transformation. (6)

2. (a) Answer any **ONE**
   i. Define the minimal polynomial of a square matrix $A$. Show that - (8)
      (p) the minimal polynomial of a real matrix $A$ divides every polynomial which annihilates $A$.
      (q) $\lambda$ is a root of the minimal polynomial of $A$ if and only if $\lambda$ is a characteristic root of $A$.
   ii. Let $A$ be an $n \times n$ matrix having $n$ eigenvalues then prove that $A$ is similar to an upper triangular matrix with the $n$ eigen values on the diagonal of the upper triangular matrix. (8)

   (b) Answer any **TWO**
   i. Let $A_{n\times n}$ be a real matrix. Show that $\lambda$ is an eigenvalue of $A$ if and only if $(\lambda I_n - A)$ is singular. Hence or otherwise show that 0 is not an eigen value of an injective linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. (6)
   ii. Let $A$ and $B$ be $n \times n$ real matrices. Prove that characteristic polynomial of $AB =$ characteristic polynomial of $BA$. (6)
   iii. Find the characteristic polynomial and the minimal polynomial of $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$. (6)
   iv. Prove that if every non-zero vector of $\mathbb{R}^n$ is an eigenvector of $A$, then $A$ is a $n \times n$ scalar matrix. (6)

[P.T.O.]

3. (a) Answer any **ONE**

   i. Show that an $n\times n$ matrix $A$ is diagonalizable if and only if sum of dimensions of eigen spaces of $A$ is $n$. (8)

   ii. Show that characteristic roots of a real symmetric matrix are real. (8)

   (b) Answer any **TWO**

   i. Let $A$ be any $n \times n$ diagonalizable matrix. Show that (6)

      (p) For any positive integer $k$, $A^k$ is also diagonalizable.

      (q) $f(A)$ is diagonalizable where $f(t)$ is any polynomial over $\mathbb{R}$.

   ii. Show that every quadratic form $Q(x_1, x_2, \cdots x_n)$ over $\mathbb{R}$ can be reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$,$X = (x_1, x_2 \cdots, x_n)^t$, $y = (y_1, y_2, \cdots y_n)^t$ and $P$ is an $n \times n$ orthogonal matrix. (6)

   iii. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Find a non-singular matrix $P$ such that $P^{-1}AP$ is a diagonal matrix and hence find $A^{100}$. (6)

   iv. Find the rotation of coordinate axes which reduces the conic $x^2 + 2xy + y^2 - 2 = 0$ to standard form. Give its equation in the standard form in the rotated system and identify the conic. (6)

4. Answer any **THREE**

   (a) Let $A_{3\times3}$ be an upper triangular matrix whose diagonal entries are 1, 2 , 3. If $A^{-1} = aA^2 + bA + cI$, then find $a, b, c$. (5)

   (b) If $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\alpha(x, y) = (ax + by + e, cx + dy + f)$ for real numbers $a, b, c, d, e, f$ is an isometry, then prove that $a^2+c^2 = 1$, $b^2+d^2 = 1$ and $ab+cd = 0$. (5)

   (c) Find the eigenvalues and the bases of the corresponding eigen spaces for a $3 \times 3$ matrix $A$ having all its entries equal to 1 . (5)

   (d) Let $V$ be a finite dimensional inner product space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation. Define an invariant subspace under $T$. Prove that for $\lambda \in \mathbb{R}$, (5)

      (i) $E_\lambda = \{X \in V / T(X) = \lambda X\}$ is a subspace of $V$.

      (ii) $E_\lambda$ is an invariant subspace under $T$.

   (e) Let $A$ and $B$ be positive definite matrices. Show that $A + B$ is also positive definite. Is the converse true? Justify your answer. (5)

   (f) Find the rank and signature of $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. (5)

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