Duration: $2\frac{1}{2}$ Hours **OLD COURSE** **Max. Marks : 75**

1) **All questions are compulsory**
2) **Figures to the right indicate marks.**

Q.1 (a) Attempt any ONE of the following **(8)**

(i) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. If P,Q are partitions of $[a,b]$ then show that $(i)\ L(P,f)\le U(P,f)\ \ ii)\ L(P,f)\le U(Q,f)$.

(ii) If f is Riemann integrable on [a,b] and a < c < b then show that f is Riemann integrable on [a,c] and [c,b] and further $\int_a^b f=\int_a^c f+\int_c^b f$.

(b) Attempt any TWO of the following **(12)**

(i) Prove that if f :[a,b]$\to\Re$ is Riemann integrable then $|f|$ is Riemann integrable. Is the converse true? Justify.

(ii) Using Riemann Criterion, show that $f:[0,3]\to\mathbb{R}$ defined by $f(x)=[x]$ is Riemann integrable on [0,3] where [x] is the integral part of x.

(iii) Let f : [0,1]$\to\Re$ defined by $f(x)=x^3$.Let $\{P_n\}$ be a sequence of partitions, given by $P_n=\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\right\}$. Calculate $U(P_n,f)$,$L(P_n,f)$ and show that $\lim_{n\to\infty}U(P_n,f)=\lim_{n\to\infty}L(P_n,f)$ and hence find $\int_0^1 f(x)dx$.

(iv) Express the following sum as a Riemann sum of a suitable function and evaluate $\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}}$

Q.2 (a) Attempt any ONE of the following **(8)**

(i) Define triple integral of a bounded function $f:Q\to\mathbb{R}$ where $Q=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ is a rectangular box in $\mathbb{R}^3$. Further show that

$$m(b_1-a_1)(b_2-a_2)(b_3-a_3)\le\iiint_Q f\le M(b_1-a_1)(b_2-a_2)(b_3-a_3)$$

where $m,\ M$ are the infimum and the supremum of f on Q. Also evaluate $\int_0^1\int_0^{2z}\int_0^{z+2}yz\,dx\,dy\,dz$.

(ii) State & prove Fubini's theorem for a rectangular domain in $\Re^2$.

(b) Attempt any TWO of the following **(12)**

(i) Use suitable change of variables to show that $\iint_S f(xy)dxdy=\log 2\int_1^2 f(u)du$ where S is the region in the first quadrant bounded by the curve xy = 1, xy = 2, y = x & y = 4x .

(ii) Find the area enclosed by one loop of four leaved rose $r=cos2\theta$.

(iii) Find the volume of the cylinder with base as the disc of unit radius in the xy-plane centred at (1,1,0) and the top being the surface $z=[(x-1)^2+(y-1)^2]^{3/2}$

(iv) Evaluate $\iiint_S(x+y+z)dxdydz$ where S is the parallelepiped bounded by the planes x+y+z=1 & x+y+z=2 , x-y+z=2 & x-y+z=3 and x-y-z=3 & x-y-z=4.

Q.3 (a) Attempt any ONE of the following **(8)**

(i) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\mathbb{R}$ .If $\{f_n\}$ converges uniformly to a function f on S then show that f is continuous on S. Further show that

$$\lim_{n\to\infty}\lim_{x\to p} f_n(x) = \lim_{x\to p}\lim_{n\to\infty} f_n(x) \text{ for each } p \in S$$

(ii) Let $\{f_n\}$ be a sequence of Riemann integrable function on [a,b].If the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on [a,b] then show that f is Riemann integrable on [a, b] and $\int_a^b\left(\sum_{n=1}^{\infty} f_n\right)dx = \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)dx\right)$.

(b) Attempt any TWO of the following **(12)**

(i) Show that the sequence $f_n(x) = \frac{x}{n}e^{-x/n}$ does not converge uniformly on $[0,\infty]$ but converge uniformly on $[0,a]$ where $a > 0$.

(ii) Show that the series $\sum_{n=0}^{\infty}\frac{(-1)^n n + x^n}{n^2}$ converges uniformly on [-1, 1].

(iii) Examine whether $\int_0^1\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right]dx =$ $\sum_{n=1}^{\infty}\int_0^1\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right]dx$. Is the series $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right]$ uniformly convergent on [0,1] ? Justify .

(iv) (iv) If a real power series $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence r, then show that it converges uniformly in $[-s,s]$ where $0 \le s < r$. Further show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in $(-r,r)$, then f is differentiable and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Q.4 Attempt any THREE of the following **(15)**

(i) If a function f defined on [a, b] is continuous and non-negative. If f(c) > 0 for some c $\in [a,b]$. Show that $\int_a^b f(x)dx > 0$.

(ii) State Riemann's criterion for integrability of a bounded function defined on [a,b] and use it to prove that the function $f(x) = x, x \in [0,1]$ is Riemann integrable .

(iii) Consider the triple integral $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} dz\,dy\,dx$. Rewrite the integral as an equivalent iterated integrals in five other ways.

(iv) Find the volume bounded by the cylinders $y^2 = x$, $x^2 = y$ and the planes z =0, x+y+z=2

(v) Let $f_n(x) = x^n$ for $x \in [0,1]$. Find $f(x) = \lim_{n\to\infty} f_n(x)$.
Show that $\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$ but $\{f_n\}$ does not converge uniformly to f on $[0,1]$.

(vI) Show that the series $\sum\{\frac{x}{[(n-1)x+1][nx+1]}\}$; x $\in [a,b]$ converge uniformly, where a > 0