**(REVISED COURSE)**

Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. Define a self complementary graph. If $G$ is self complementary graph of order $p$, show that $G$ is connected and $p \equiv 0$ or $1( \mod 4)$
   ii. State and prove $Havel - Hakimi$ theorem for degree sequence of a graph $G$.

   (b) Attempt any **TWO** questions: (12)
   i. Define adjacency matrix of a graph G. If $(A^n) = (a_{ij}^n)$ is the $n^{th}$ power of adjacency matrix $A$ of a graph $G$ with $V(G) = \{v_1, v_2, \ldots v_n\}$, then show that the number of triangles in $G$ is $\frac{1}{6}$trace of $A^3$.
   ii. If $G$ is graph of order n with $\delta(G) \geq (n-1)/2$, then show that G is connected. Is the bound $(n-1)/2$ sharp?, that is, in this case, can $(n-1)/2$ be replaced by$(n-2)/2$ ?
   iii. Show that every $u - v$ walk $W$ contains $u - v$ path.
   iv. If $G$ and $H$ are isomorphic graphs, then show that the degree sequence of the vertices of $G$ are the same as the degree sequence of the vertices of $H$.

2. (a) Attempt any **ONE** question: (8)
   i. Let $G$ be a $(p, q)$ graph. Prove that following statements are equivalent.
      1) $G$ is tree.
      2) $G$ is acyclic and $q = p - 1$.
      3) $G$ is connected and $q = p - 1$.
   ii. Define a spanning tree of a graph $G$. Show that a graph is connected if and only if it has a spanning tree.

   (b) Attempt any **TWO** questions: (12)
   i. Show that each label spanning tree with $n$ vertices corresponds to a unique vector $s = (s_1, s_2, ....s_{n-2})$ where $s_i \in \{1, 2, ..., n\}$ for $i = 1, 2, ..., n$
   ii. Let $\tau(G)$ denote the number of spanning trees of a graph $G$. If $e \in E(G)$ is not a loop, then prove that $\tau(G) = \tau(G - e) + \tau(G.e)$.
   iii. If $T$ is spanning tree of a connected graph $G$ and $e$ is an edge of $G$ that is not in $T$, then prove that $T + e$ contains a unique cycle that contains the edge $e$.
   iv. Describe the trees produced by Breath First Search (BFS) and Depth First Search (DFS) algorithm for the complete graph $K_n$ where $n$ is positive integer. Justify your answer.

**[TURN OVER**

3. (a) Attempt any **ONE** question: (8)

   i. Prove that a connected graph G contains Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, prove that each Eulerian trail of G begins at one of these odd vertices and ends at the other.

   ii. If $u$ and $v$ are non-adjacent vertices in a graph $G$ such that $deg(u) + deg(v) \geq p$, then show that $G$ is Hamiltonian if and only if $G + uv$ is Hamiltonian

   (b) Attempt any **TWO** questions: (12)

   i. Define closure $C(G)$ of a graph $G$. Show that a simple graph is Hamiltonian if and only if its closure is Hamiltonian.

   ii. Prove that the cube graph $Q_k$ is bipartite $k-$regular graph with $2^k$ vertices.

   iii. If $G$ is Hamiltonian graph then for every nonempty proper subset $S$ of $V(G)$, prove that $\omega(G - S) \leq |S|$. Give an example of a graph which satisfies the above condition but not Hamiltonian.

   iv. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the $27 \times 1 \times 1$ sub-cubes. If he starts at one corner and always move on to an uneaten sub-cube, can he finish at the centre of the cube?

4. Attempt any **THREE** questions: (15)

   (a) Prove that every $(p, q)$ graph with $q \geq p$ contains a cycle. Is it true if $q \geq p - 1$? Justify.

   (b) Explain Dijkstra's algorithm to find shortest path in a graph $G$.

   (c) Prove that if $G$ is a connected graph of order $p \geq 3$ and $G$ has a cut edge then $G$ contains a cut vertex. Is the converse true? Justify.

   (d) Describe Kruskal's algorithm for finding minimum spanning tree in a connected weighted graph.

   (e) Define a line graph of a graph $G$. Show that the line graph a simple graph $G$ is a path if and only if $G$ is a path.

   (f) Show that complete bipartite graph $K_{n,n}$ is Hamiltonian.

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