T.Y.B.Sc Sem. V

Subject: Mathematics

Integral Calculus (Rev). (R-2016)-17

QP Code : 77109

Max Marks:75]

Revised Course

Duration: 21/2 Hours

N.B. 1. All questions are compulsory.
2. From Question 1,2 and3, attempt any one from part(a), any two from part(b).
3. From Question 4, Attempt any THREE
4. Figures to the right indicate marks for the respective parts.

Q.1 a i Prove that a continuous function is integrable for a rectangular domain in $\mathbb{R}^2$. $f(x,y)=\begin{cases}0 & if\ x,y\in\mathbb{Q}\cap R\\ 3 & if\ otherwise\end{cases}$ where $R=[0,1]\times[0,1]$. Is f integrable over R ? Justify your answer. (8)

ii Let U be an open set in $\mathbb{R}^2$ containing the rectangle $[a,b]\times[c,d]$. Suppose $f:U\to\mathbb{R}$ is a continuously differentiable function. Show that $g'(x)=\int_c^d\frac{\partial f}{\partial x}(x,y)dy$ where $g(x)=\int_c^d f(x,y)dy, \forall\, x\in[a,b]$.

b i State the change of variables formula for triple integral, stating clearly the conditions under which it is valid. Use it to express $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dzdydx$ in spherical co-ordinates $(\rho,\theta,\emptyset)$. (12)

ii Evaluate $\iint_S (x^2+y^2)\,dxdy$ where S is the region in the XY-plane bounded by the curves $x^2-y^2=1, x^2-y^2=2, xy=2, xy=4$ by using a suitable change of variable.

iii Evaluate $\int_0^2\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dydx$ by converting to polar coordinates.

iv Find the centre of mass of a thin plate in the shape of a rectangle ABCD if the density at any point is the product of the distances of the point from two adjacent sides AB and AD.

Q.2 a i Suppose F is a continuous vector field defined on an open connected set U in $\mathbb{R}^n$. Define a function $\phi:U\to\mathbb{R}$ by $\phi(v)=\int_{v_0}^{v}F$ where v_0 is a fixed point in U and F is conservative. Show that $\nabla\phi(v)=F(v)\ \forall\, v\in U$ (8)

ii State and prove Green's Theorem for a rectangle. If R is the region enclosed between the circles $x^2+y^2=1$ and $x^2+y^2=4$, express $\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$ in terms of line integral.

b i If U is an open set in $\mathbb{R}^n$ and Γ is a parameterized curve in U, define the line integral of f along Γ for a continuous function $f:U\to\mathbb{R}$. Show further, if α,β are two equivalent parameterized curves, then the line integrals of f along them coincide. (12)

ii Evaluate $\int_C F$, where $F(x,y,z)=\sin z i+\cos\sqrt{y}j+x^3k$ and C is the line segment from $(1,0,0)$ to $(0,0,3)$.

iii Consider the vector field $\vec{F}(x,y,z)=e^y\vec{i}+xe^y\vec{j}+(z+1)e^z\vec{k}$. Find a scalar function f such that $\vec{F}=\nabla f$. Hence evaluate the line integral $\int_C\vec{F}$ where C is given by $\vec{r}(t)=t\vec{i}+t^2\vec{j}+t^3\vec{k}$ for $0\le t\le 1$.

iv u,v are two scalar fields which are continuously differentiable on $\mathbb{R}^2$. Define $f(x,y)=(v,u)\,,g(x,y)=(ux-uy,vx-vy)$. Find $\iint_S f.g$ if $u(x,y)=1$ and $v(x,y)=y$ on the boundary of the unit disk S.

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Q.3 a i Let $S = \bar{r}(T)$ be a smooth parametric surface described by a differentiable function $\bar{r}$ defined on region T. Let f be defined and bounded on S. Define surface integral of f over S. If R and r are smoothly equivalent functions, $R(s,t) = \bar{r}(G(s,t))$ where $G(s,t) = u(s,t)\hat{\imath} + v(s,t)\hat{\jmath}$ being continuously differentiable. Then show that $\iint_{r(A)} f dS = \iint_{R(B)} f dS$ where $G(B) = A$. (8)

ii State and Prove Divergence Theorem for a simple solid region V bounded by an orientable surface S which can be projected on XY, YZ, ZX planes.

b i $f(x,y,z) = z^2$ and S is the portion of the cone $x^2 + y^2 = z^2$ between the plane $z = 1$ and $z = 2$. Evaluate surface integral of the scalar field f over S (1

ii Evaluate surface integral of the vector field $F(x,y,z) = (18z, -12, 3y)$ over S which is the surface of $2x + 3y + 6z = 12$ in the first octant.

iii Using Stokes' Theorem evaluate the line integral of $F(x,y,z) = (x^2, y^2, z^2)$ along C where C is the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.

iv For the surface $\bar{r}(u,v)$ described by the vector equation $\bar{r}(u,v) = X(u,v)i + Y(u,v)j + Z(u,v)\hat{k}$, $(u,v) \in T$ where X, Y, Z are differentiable on T, define the fundamental vector product $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$. If C is a smooth curve lying on the surface, $C = \bar{r}(\alpha(t))$, $\alpha: [a,b] \to T$, then show that $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ is normal to C at each point.

Q.4 a) $f(x,y) = x + y$ and S is the region bounded by the parabola $y = x^2$ and $y = 1 - x^2$. Sketch the region S of integration. Express $\iint_S f$ in terms of both the iterated integrals. (1

b) Using cylindrical co-ordinates, evaluate the integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} x^2 dzdydx$$

c) $\bar{F} = \frac{-y\,\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$. Is F conservative ? Justify your answer.

d) $F = (P,Q)$ is a continuously differentiable function defined on a open, simply connected region D in $\mathbb{R}^2$. Show that F is conservative on D if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x,y) \in D$.

e) Assume S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second order partial derivatives. Prove with usual notations that $\int_C (f\nabla g).d\bar{r} = \iint_S (\nabla f \times \nabla g).\hat{n}dS$

f) Find the surface area of S where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.