2 $\frac{1}{2}$ Hours] **(Revised Syllabus)** **[Total Marks: 75**

N.B.: (1) **All questions are compulsory**.
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is such that (8)
      (i) $T(0) = 0$.
      (ii) $\|T(x) - T(y)\| = \|x - y\|$, for all $x, y \in \mathbb{R}^n$.
      then, show that T is an orthogonal linear transformation.
   ii. State and prove the Cayley Hamilton theorem. (8)

   (b) Answer any **TWO**
   i. Show that any orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation about origin or a reflection about a line passing through origin. (6)
   ii. Let V be a finite dimensional inner product vector space. Let $T : V \to V$ be a linear transformation. Prove that T is orthogonal if and only if $\|T(x)\| = \|x\|, \forall\, x \in V$. (6)
   iii. Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (\frac{1}{2}x + \frac{\sqrt{3}}{2}z - 1, y, , \frac{\sqrt{3}}{2}x - \frac{1}{2}z + 5)$ is an isometry. Express it as a composition of an orthogonal transformation and a translation map. (6)
   iv. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Find the characteristic polynomial of A and using Cayley-Hamilton theorem find A^{-1}. (6)

2. (a) Answer any **ONE**
   i. Show that a real square matrix $A_{n\times n}$ with n real eigen values is similar to an upper triangular matrix. (8)
   ii. Define minimal polynomial of a square matrix. Show that the minimal polynomial of a real square matrix A divides every polynomial that annihilates $A_{n\times n}$. (Polynomial with real coefficients $f(x)$ annihilates A if $f(A) = 0$). (8)

   (b) Answer any **TWO**
   i. Define eigen value of a real square matrix. Prove that $\lambda \in \mathbb{R}$ is an eigen value of $A_{n\times n}$ if and only if λ is a root the characteristic polynomial. (6)
   ii. Show that eigen vectors $v_1, v_2, \cdots, v_k$ corresponding to distinct eigen values $\lambda_1, \lambda_2, \cdots, \lambda_k$ respectively of a square matrix A are linearly independent. (6)
   iii. Let V be a vector space of dimension 3 and $\{v_1, v_2, v_3\}$ be a basis of V. Find eigen values and corresponding eigen spaces of $T : V \to V$ be defined by $T(v_1) = v_1, T(v_2) = v_1 + v_2, T(v_3) = v_1 + v_2 + v_3$. (6)
   iv. Let $A_{n\times n}$ be a real matrix. if A has n distinct characteristic roots, then prove that the characteristic polynomial of A = the minimal polynomial of A. (6)

[P.T.O.]

3. (a) Answer any **ONE**

   i. Show that a matrix is orthogonally diagonalizable if and only if there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigen vectors of A. (8)

   ii. Show that if λ is eigen value of real symmetric matrix A, then $\lambda \in \mathbb{R}$. Also prove that the eigen vectors associated with distinct eigen values of A are orthogonal. (8)

   (b) Answer any **TWO**

   i. Show that algebraic multiplicity of an eigen value of a square matrix is greater than or equal to its geometric multiplicity. (6)

   ii. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix and hence find A^{100}. (6)

   iii. Let A is diagonalizable matrix such that eigen values of A are (6)
   (p) ± 1, then show that A is invertible and $A = A^{-1}$.
   (q) 0 and 1 then show that $A^2 = A$.

   iv. Show that every quadratic form $Q(x_1, x_2, \cdots x_n)$ over $\mathbb{R}$ can be to reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$,$X = (x_1, x_2 \cdots, x_n)^t$, $y = (y_1, y_2, \cdots y_n)^t$ and P is an $n \times n$ orthogonal matrix. (6)

4. Answer any **THREE**

   (a) Find an orthogonal transformation in $\mathbb{R}^3$ which represents reflection with respect to the plane $x - 2y + z = 0$. (5)

   (b) Let $V = M_2(\mathbb{R})$ and W = Space of 2×2 real symmetric matrices. Find a basis of W and the quotient space V/W. (5)

   (c) Let $A_{n\times n}$ be an orthogonal matrix with $det\ A = -1$, then show that -1 is an eigen value of A. (5)

   (d) Find the minimal polynomial of the diagonal matrix $A = \{1, -1, 1, -1\}$. (5)

   (e) Show that $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $a, b, d \in \mathbb{R}$ is diagonalizable if and only if $b = 0$ or $a \neq d$. (5)

   (f) Let $A = \begin{pmatrix} \alpha & -3 \\ 3 & 0 \end{pmatrix}$, $\alpha \in \mathbb{R}$ is a parameter, then show that A is diagonalizable if $-6 < \alpha < 6$. (5)

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