2 $\frac{1}{2}$ Hours] **(Old Syllabus)** **[Total Marks: 75]**

N.B.: (1) **All questions are compulsory**.
(2) Figures to the right indicate marks for respective subquestions.

1. (a) Answer any **ONE**
   i. State and prove the first isomorphism theorem of vector space. (8)
   ii. Let V be a finite dimensional inner product space over $\mathbb{R}$. If $f : V \to V$ is a map such that (i) $f(0) = 0$ (ii) $\|f(x) - f(y)\| = \|x - y\| \quad \forall x, y \in V$, then show that f is an orthogonal linear transformation. (8)

   (b) Answer any **TWO**
   i. Let W be a subspace of vector space V, define V/W. Show that following operations are well defined on V/W for $x, y \in V$ and $\alpha \in \mathbb{R}$. $W+x \oplus W+y = W+(x+y)$ and $\alpha \odot W+x = W+\alpha x$. (6)
   ii. Define orthogonal linear transformation. For finite dimensional inner product space V, $T : V \to V$ is a linear transformation, prove that following statements are equivalent : (6)
      (p) T is orthogonal.
      (q) If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V then $\{T(e_i)\}_{i=1}^n$ is also an orthonormal basis of V.
   iii. Show that a 2×2 orthogonal matrix A with $det\ A = -1$ is a matrix of reflection about a line passing through origin. (6)
   iv. Let $V = M_2(\mathbb{R})$, $W = \{A \in M_2(\mathbb{R}) : Tr(A) = 0\}$ be a subspace of V. Find bases of W and V/W. (6)

2. (a) Answer any **ONE**
   i. Show that an $n \times n$ real matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigen vectors of A. (8)
   ii. Show that every real symmetric matrix of order n is orthogonally diagonalizable. (8)

   (b) Answer any **TWO**
   i. Show that the characteristic roots of a real symmetric matrix are real. (6)
   ii. Let $A_{n\times n}$ be a real symmetric matrix. Prove that $AX \cdot Y = X \cdot AY$ for every $X, Y \in \mathbb{R}^n$ regarded as column vectors. Hence or otherwise prove that eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal. (6)
   iii. Let $A = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$. Find a non-singular matrix P s.t. $P^{-1}AP$ is a diagonal matrix and find A^{100}. (6)
   iv. Define positive definite real symmetric matrix. Show that, if $A_{n\times n}$ is a positive definite real symmetric matrix then all the eigen values of A are positive. (6)

[P.T.O]

3. (a) Answer any **ONE**

   i. Let G be a cyclic group of order n generated by a. Prove that a^m generates G if and only if gcd $(m, n) = 1$. (8)

   ii. State and prove Lagrange's theorem. (8)

   (b) Answer any **TWO**

   i. Show that an infinite cyclic group has only two generators. (6)

   ii. Prove that if H and K are subgroups of a group G then HK is a subgroup of group G if and only if $HK = KH$. (6)

   iii. For a group G, $(ab)^3 = a^3b^3$, $(ab)^4 = a^4b^4$, $(ab)^5 = a^5b^5$ for all $a, b \in G$ then show that G is abelian. (6)

   iv. Let $f : G \to G'$ be a group homomorphism. Prove that : (6)

   p. $f(e) = e'$, where e, e' are identity elements of G and G' respectively.

   q. $f(a^{-1}) = (f(a))^{-1}$, $\forall a \in G$.

   r. $f(a^n) = (f(a))^n$, $\forall n \in \mathbb{N}$.

4. Answer any **THREE**

   (a) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & -3 & 0 \\ 5 & 2 & 0 \end{pmatrix}$. Using the Cayley-Hamilton theorem, find $A^4 + A^3 - 5A^2 + A + 2I_3$. (5)

   (b) Let A be 3×3 orthogonal matrix such that det $A = -1$. Show that -1 is an eigen value of A. (5)

   (c) Show that an $n \times n$ matrix A having n distinct eigen values is diagonalizable. (5)

   (d) Define rank and signature of the quadratic form $Q[X]$. Find the rank and signature of $Q[X] = 2x^2 + 2y^2 - 2xy$. (5)

   (e) Show that the set $\{\bar{5}, \bar{15}, \bar{25}, \bar{35}\}$ under multiplication modulo 40 is a group. (5)

   (f) Show that the group $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \right\}$ is isomorphic to the group $\mathbb{C} - \{0\}$ of non-zero complex numbers under multiplication. (5)

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