Duration 2 $\frac{1}{2}$Hrs | OLD COURSE | Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   - (i) Let (X, d) be a metric space. Define limit point of $F \subseteq X$. Also show that F is closed if and only if F contains all it's limit points.
   - (ii) In a metric space (X, d) , prove that arbitrary union of open sets is open in X. Give an example to show that arbitrary intersection of open sets is not open in X.

   (b) Attempt any Two questions: (12)
   - (i) Show that $U = \{(x, y) \in \mathbb{R}^2 : \ 2x + 3y < 1\}$ is an open subset of $\mathbb{R}^2$ with Euclidean metric.
   - (ii) Let (X, d) be a metric space and $A \subseteq X$. Show that A° is an open set and is the largest open set contained in A.
   - (iii) Prove that in any metric space (X, d), $A \subseteq X$, A is closed if and only if $\partial A \subseteq A$ where ∂A denotes the boundary of A.
   - (iv) State and prove Hausdorff property in a metric space (X, d)

2. (a) Attempt any one question: (8)
   - (i) If in a metric space (X, d), for every decreasing sequence $\{F_n\}$ of non-empty closed sets with $d(F_n) \longrightarrow 0$, we have $\cap_{n\in\mathbb{N}} F_n$ is a singleton set then prove that (X, d) is complete.
   - (ii) Let (X, d) be a metric space and $A \subseteq X$. Prove that $p \in \overline{A}$ if and only if there is a sequence of points in A converging to p.

   (b) Attempt any Two questions: (12)
   - (i) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2).
   - (ii) Check if Cantors Theorem is applicable in the following examples. Also , find $\bigcap_{n\in\mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual.
     (I) $F_n = [n, \infty)$
     (II) $F_n = (0, \frac{1}{n})$
   - (iii) Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.
   - (iv) Show that a sequence (x_n) in $(\mathbb{R}^2, d)$ (where d is Euclidean distance) converges to a point $p = (p_1, p_2) \in \mathbb{R}^2$ if and only if $({x_n}^i) \longrightarrow p_i$ for $1 \le i \le 2$, in $\mathbb{R}$ with respect to the usual distance, where $x_n = ({x_n}^1, {x_n}^2)$.

3. (a) Attempt any One from the following: (8)

   (i) Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Show that f is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.

   (ii) Let (X, d) and (Y, d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous on X if and only if for each subset A of X, $f(\overline{A}) \subseteq \overline{(f(A))}$

   (b) Attempt any Two questions: (12)

   (i) If $f, g : (X, d) \longrightarrow (Y, \rho)$ are continuous on X and $f(x) = g(x)\ \forall x \in A,\ A \subseteq X$, then show that $f(x) = g(x)\ \forall x \in \overline{A}$.

   (ii) Prove or disprove: Continuous image of of an open set is open.

   (iii) Let (X, d) and (Y, d') be metric spaces. When is $f : X \longrightarrow Y$ said to be uniformly continuous? Show that $f(x) = \dfrac{1}{(1 + x^2)}$ is uniformly continuous on $\mathbb{R}$ (under usual metric).

   (iv) Prove every function $f : (\mathbb{N}, d_{\text{usual}}) \longrightarrow (Y, d)$ where (Y, d) is any metric space is continuous.

4. Attempt any Three questions: (15)

   (a) Show that $U = \{(x, y) \in \mathbb{R}^2 : \ 2x + 3y < 1\}$ is an open subset of $\mathbb{R}^2$ with Euclidean metric.

   (b) Show that $d : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ is a metric on $\mathbb{N}$ where d is defined as follows:

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{(m + n)} & \text{if } m \neq n. \end{cases}$$

   (c) Let $X = C[0, 1]$ and d_1 be the metric induced by $\|\ \|_1$ on X. ($\|f\|_1 = \int_0^1 |f(t)|dt$). Show that the following sequence of functions $\{f_n\}$ is bounded in (X, d_1)

$$f_n(t) = \begin{cases} 8n^2 t & \text{if } \ 0 \leq t \leq \frac{1}{4n} \\ -8n^2 t + 4n & \text{if } \ \frac{1}{4n} < t \leq \frac{1}{2n} \\ 0 & \text{if } \ \frac{1}{2n} < t \leq 1 \end{cases}$$

   (d) Let (X, d) and (Y, d') be metric spaces. Show that if $f : X \longrightarrow Y$ is uniformly continuous on X and if (x_n) in X is Cauchy then show that the sequence $(f(x_n))$ is Cauchy in Y.

   (e) Let (X, d) be a metric space and let $A \subseteq X$, If $d_A : X \longrightarrow \mathbb{R}$ is defined by $d_A(x) = d(x, A)$. Then show that d_A is continuous on X. (distance in $\mathbb{R}$ being usual)

   (f) Prove or disprove: Continuous image of of an open set is open.

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