Duration 2 $\frac{1}{2}$Hrs REVISED COURSE Marks: 75

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) In a metric space (X, d), prove that arbitrary union of open sets is open in X. Give an example to show that arbitrary intersection of open sets is not open in X.
   (ii) Let (X, d) be a metric space and $A, B \subseteq X$. Show that
   (I) $A \subseteq B \Longrightarrow A^\circ \subseteq B^\circ$
   (II) $(A \cap B)^\circ = A^\circ \cap B^\circ$
   (III) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

   (b) Attempt any Two of the following: (12)
   (i) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2).
   (ii) $(\mathbb{Z}, d)$ and $(\mathbb{Z}, d_1)$ are metric spaces where d is the usual metric (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{Z}$. Prove that d and d_1 are equivalent metrics.
   (iii) Let d_1, d_2 be metrics on X. Define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x, y) = \max \{d_1(x, y), d_2(x, y)\}$. Show that d is a metric on X.
   (iv) $\| \ \|_1$ and $\| \ \|_2$ are norms on $\mathbb{R}^2$ where for $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\|_1 = \sum_{i=1}^{i=2} |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^{i=2} x_i^2}$. Show that $\|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{2}\, \|x\|_2$ for $x \in \mathbb{R}^2$

2. (a) Attempt any One of the following: (8)
   (i) Show that for a subset F of a metric space (X, d), the following statements are equivalent:
   (I) F is closed
   (II) F contains all its limit points.
   (ii) Let (X, d) be a metric space and A be a subset of X. Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p .

   (b) Attempt any Two of the following: (12)
   (i) Let $A, B \subset \mathbb{R}$ (distance being usual), where $A = \mathbb{N}$ and $B = \left\{ n + \frac{1}{n} : n \in \mathbb{N}, n > 1 \right\}$. Find $d(A, B)$
   (ii) Let A be a subset of a metric space (X, d) . Prove that
   (I) $\overline{(X \setminus A)} = X \setminus A^\circ$
   (II) $(X \setminus A)^\circ = X \setminus (\overline{A})$
   .
   (iii) Show that $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a closed subset of $\mathbb{R}^2$, where the distance being Euclidean .
   (iv) Prove that a subset A of a metric space (X, d) is dense in X if and only if $G \cap A \neq \emptyset$ for each non-empty open subset G of X.

3. (a) Attempt any One of the following: (8)

(i) If $K \subseteq \mathbb{R}^n$ is such that K is compact then prove that K has Bolzano-Weierstrass Property. (distance being Euclidean).

(ii) Show that a compact subset of $(\mathbb{R}^n, d)$ where d Euclidean, is closed and bounded . Give an example to show that a closed and bounded subset of a metric space is not compact.

(b) Attempt any two: (12)

(i) Prove that a subset of a discrete metric space is compact if and only if it is finite.

(ii) (X, d) is a metric space and (x_n) is a sequence in X such that (x_n) converges to some point $p \in X$. If $S = \{x_n : n \in \mathbb{N}\} \cup \{p\}$ then show that S is compact by using the definition of compactness.

(iii) Let A, B be compact subsets of $(\mathbb{R}, d)$, distance d being usual. Show that $A \times B$ is a compact subset of $(\mathbb{R}^2, d')$ where d' is the Euclidean distance.

(iv) Consider the metric space $(\mathbb{R}, d)$, where d is the usual distance . Show that $\{(\frac{1}{n}, \ 1) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. Is $(0, 1)$ compact ? Justify your answer.

4. Attempt any Three of the following: (15)

(a) Prove or disprove : If (X, d) be a metric space and $x, y \in X, r, s > 0$ and $B(x, r) = B(y, s)$, then either $x = y$ or $r = s$

(b) Show that $\| \ \|$ is a norm on X, where $X = M_2(\mathbb{R})$ and $\|A\| = \max \{|a_{ij}| : 1 \le i, j \le 2\}$ for $A = (a_{ij}) \in X$

(c) Let (X, d) be a discrete metric space and $A \subseteq X$. Then prove that $\overline{(A^\circ)} = \overline{A}$

(d) Consider the sequence (f_n) of functions in $C[0, 1]$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \dfrac{n}{2} + 1 & \text{if} \quad \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < t \le 1 \end{cases}$$

Show that $\{f_n\}$ is Cauchy w.r.t. $\| \ \|_1$ where $\|f\|_1 = \displaystyle\int_0^1 |f(t)| dt$

(e) Let $A = \{(x, y) \in \mathbb{R}^2 : |x| \le 1\}$. Determine whether A is compact. Justify your answer.

(f) Prove or disprove :] A closed ball $B[x, r]$ in a metric space is compact.