(3 Hours) **[Total Marks : 100]**

N.B. **1**. All questions are compulsory.

2. Figures to the right indicate marks for respective parts

3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. Which of the following equation has a root in (1,2)?

(a) $4x^2 - 1 = 0$ (b) $sinx - x = 0$

(c) $2x^3 - 3x^2 + 2x - 3 = 0$ (d) None of these

ii. If A is countable set and B is uncountable set then the most we can say about A ▯ B is _____.

(a) Finite (b) At most countable

(c) Uncountable (d) Countable

iii. If the decimal representation of a number is non terminating and non repeating then the number is ______.

(a) A whole number (b) An irrational number

(c) A natural number (d) A rational number

iv. The norm of the partition P = { −6, −5.8, −5.2, −4.8, −3.2, −3 } is

(a) 2.6 (b) 1.4

(c) 1.6 (d) None of these

v. Let P and Q be any two Partitions of interval $[a, b]$. Then the statement that is always true is

(a) $L(P, f) \leq U(Q, f)$ (b) $U(P, f) \leq U(Q, f)$

(c) $U(P, f) \leq L(Q, f)$ (d) None of these

vi. Let f *and* g be functions such that the function $f + g$ is integrable on I, then

(a) Both f and g must be integrable on I

(b) At least one of f and g must be integrable on I

(c) f and g may or may not be integrable on I

(d) None of these

vii. Let $F: [\pi, 2\pi] \to \mathbb{R}$ such that $F(x) = \int_{\pi}^{x} \sin^2 t dt$ then $F'(x) = \cdots\cdots$

(a) $\sin^2 x$

(b) $\cos^2 x$

(c) $cosx\ sinx$

(d) $2sinx$

viii. Let f(x) and g(x) be two positive R-integrable functions on $[a, \infty]$ such that $f(x) \le g(x), \forall x \in [a, \infty]$ then

(a) $\int_a^\infty g(x)dx$ is convergent if $\int_a^\infty f(x)dx$ is convergent

(b) $\int_a^\infty g(x)dx$ is divergent if $\int_a^\infty f(x)dx$ is divergent

(c) $\int_a^\infty f(x)dx$ is divergent if $\int_a^\infty g(x)dx$ is divergent

(d) both(a) and (b) are true.

ix. $\Gamma(n+1)$ is

(a) $(n+1)!$

(b) $(n+1)$

(c) $n!$

(d) n

x. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ represents

(a) ellipsoid

(b) sphere

(c) Hyperboloid of one sheet

(d) Hyperboloid of two sheets

Q.2 a) Attempt any ONE question from the following: (08)

i. State and prove Nested Interval Theorem.

ii. Using Nested Interval Theorem prove that if $f: [a, b] \to \mathbb{R}$ is a continuous function with $f(a)f(b) < 0$ then there exists $c \in (a, b)$ s.t. $f(c) = 0$.

b) Attempt any TWO questions from the following: (12)

i. If $I_n = \left(0, \frac{1}{n}\right)$ for all n∈N then prove that $\bigcap_{n=1}^{\infty} I_n = \emptyset$

ii. Show that a real number is rational iff it has repeating decimal representation.

iii. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$

iv. Find convergent subsequence of sequence $\sin(\frac{n\pi}{2})$.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is R-integrable on $[a, b]$ iff for any $\in > 0$, there exists a partition $P_\in$ of $[a, b]$ such that $U(f, P_\in) - L(f, P_\in) < \in$.

ii. Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Then show that f is Riemann integrable on $[a, b]$.

b) Attempt any TWO questions from the following: (12)

i. Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $m = Inf\ (f)$ and $M = Sup(f)$ on $[a, b]$. With usual notations, define $L(P, f)$ and $U(P, f)$ where P is a partition of $[a, b]$. Hence prove that $m\,(b - a) \leq L(P, f) \leq U(P, f) \leq M\,(b - a)$.

ii. Prove that the function : $[0, 4] \longrightarrow \mathbb{R}$ defined by $f(x) = 2x^2 + 1$ is Riemann integrable and evaluate $\int_0^4 f(x)dx$.

iii. Let $f : [3, 6] \longrightarrow \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 12 & for\ 3 \leq x < 6 \\ 500 & for\ x = 6 \end{cases}$$
Prove that f is Riemann integrable and $\int_3^6 f(x)dx = 36$.

iv. Using Riemann Criterion, show that the function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is Riemann integrable.

Q.4 a) Attempt any ONE question from the following: (08)

i. Let $f: [a, b] \to \mathbb{R}$ be R-integrable on $[a, b]$ and $F(x) = \int_a^x f(t)dt, \forall x$. If f is continuous on $[a, b]$ then show that F is differentiable and $F'(x) = f(x)$

ii. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff $m > 0, n > 0$.

b) Attempt any TWO questions from the following: (12)

i. Solve the improper integral $\int_{-1}^{1} \frac{1}{x^2} dx$.

ii. By using "integration by parts" solve the integral $\int_0^2 xe^x dx$.

iii. Express $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$ in terms of beta function.

iv. Evaluate $\iint_D ydA$ where D is the region bounded by the line $y = x$ and the parabola $y = 4x - x^2$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Show that $G = \{\left(\frac{1}{n+4}, \frac{1}{n}\right) /\ n \in N\}$ covers $A = (0,1)$ but has no finite sub-cover for A.

b) Show that set of real numbers $\mathbb{R}$ is uncountable.

c) Let P = { 0, 1, 2, 3, 4 } be a partition of [0, 4] and $f : [0, 4] \longrightarrow \mathbb{R}$ is a function such that $f(x) = 3 - x^2$ then find the lower sum $L(P, f)$ and upper sum $U(P, f)$.

d) Show that the function $f : [1, 3] \longrightarrow \mathbb{R}$ is Riemann integrable, where

$$f(x) = 5 \quad for \quad 1 \le x < 2$$

$$= -9 \quad for \quad 2 \le x \le 3$$

e) Check the convergence of improper integral $\int_2^\infty \frac{1}{\sqrt{x^2 - 1}} dx$ by comparison test.

f) Reverse the order of integration and evaluate $\int_0^1 \int_2^{4-2x} xdydx$.

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