(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.

(ii) Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ The set $S = \{(x, y) \in \mathbb{R}^2 \ / \ 0 \le x^2 + y^2 \le 3\}$ is

(a) a closed set. (b) open as well as closed set.

(c) an open set. (d) None of these.

$ii.$ Let $g: \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \dfrac{x^2 - 4y^2}{x - 2y} & if \ x \neq 2y \\ g(x,y) & if \ x = 2y \end{cases}$$

And if f is continuous on the whole plane, then $g(x, y)$ is

(a) $2xy$ (b) x

(c) $4y$ (d) None of these

$iii.$ $f(x,y) = 100 - x^2 + y^2$. Then the direction along which the directional derivative of f at $(5, 6)$ is 0, is

(a) $(-10, 12)$ (b) $(12, 10)$

(c) $(2,2)$ (d) None of these

$iv.$ Let A: Total derivative is a linear transformation.

B: Every differentiable scalar field is continuous.

Then which of the following is true?

(a) A is true, B is false. (b) A is false, B is true.

(c) Both A & B are true. (d) Both A & B are false.

$v.$ If $f(x, y) = |xy|, \forall (x, y) \in \mathbb{R}^2$ then

(a) f is differentiable at (0,0).

(b) f is continuous at (0,0) and $D_u f(0,0)$ exist for any vector u.

(c) The partial derivatives f_x, f_y does not exist at (0,0).

(d) None of these.

vi. If $f: \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\frac{\partial f}{\partial y} = 0$,then

(a) f is independent of x and z. (b) f depends on x and z only.

(c) f is constant. (d) None of these.

vii. Which of the following is the level set of $f(x, y, z) = x^2 + y^2 + z^2$ for $k = 1$?

(a) Sphere of radius 1 centered at origin. (b) Circle of radius 1 centered at origin.

(c) Sphere of radius 2 centered at origin. (d) Sphere of radius 1 centered at (1,0,0).

viii. If $u(x, y) = x^2 + y^2$, $x = r + e^s$, $y = log(s)$ then $\frac{\partial u}{\partial r}$ is

(a) $r + e^s$ (b) $2r + 2e^s$

(c) r (d) e^s

ix. A critical point of the function $f(x, y) = x^2y - x - y$ is

(a) $(1, 1)$ (b) $\left(1, \frac{1}{2}\right)$

(c) $\left(1, -\frac{1}{2}\right)$ (d) None of these

x. Saddle point is a point where

(a) the function has maximum value. (b) the function has minimum value.

(c) the function has zero value. (d) the function has neither maximum nor minimum value.

Q2. Attempt any **ONE** question from the following: (08)

a) i. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let $l \in \mathbb{R}$ such that $\lim_{(x,y) \to (a,b)} f(x, y) = l$. Also assume that the one dimensional limits $\lim_{x \to a} f(x, y)$ and $\lim_{y \to b} f(x, y)$ exists, then prove that $\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y) = l$.

ii. If $\langle x_n \rangle$ and $\langle y_n \rangle$ are convergent sequences in $\mathbb{R}^n$ and α, β are real constant, show that $\langle \alpha x_n + \beta y_n \rangle$ is also convergent in $\mathbb{R}^n$ and

$$\lim_{n\to\infty} \alpha x_n + \beta y_n = \alpha \lim_{n\to\infty} x_n + \beta \lim_{n\to\infty} y_n.$$

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. Using $\epsilon - \delta$ definition show that f is continuous at (0,0), where

$$f(x,y) = \begin{cases} x^{\frac{4}{3}} \sin\left(\frac{1}{x}\right) + y^{\frac{4}{3}} \sin\left(\frac{1}{y}\right) & if\ xy \neq 0 \\ 0 & if\ xy = 0 \end{cases}$$

ii. Prove that every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$.

iii. Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Define $D_i f(a)$, the i-th partial derivative of f at a, $1 \leq i \leq n$. Determine whether the partial derivatives of f exist at $(0,0)$ for the following function. In case they exist, find them.

$f(x,y) = \|(x,y)\|^4$

iv. Let $f: \mathbb{R}^2 \to \mathbb{R}, a = (-1,2), u = (3,-4), v = (12,5)$ and $w = (15,1)$. If $D_u f(a) = 8,\ D_v f(a) = 1$ find $D_w f(a)$.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let U be an open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be differentiable at $a \in U$. Prove that $D_i f(a)$ exists for each $i = 1,2, \ldots\ldots, n$. Explain with an example that converse of this is not true.

ii. State and prove sufficient condition for the equality of mixed partial derivatives.

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Find total derivative as linear transformation T for the function $f(x,y) = x^2 + 2xy + y^2$ at the point $a = (-1,-2)$.

ii. Find directional derivative of $f(x,y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at (1,2).

iii. Find the equation of the tangent plane and normal line to the surface $x^3 + 7x^2z + z^3 = 4$ at $(2,1,-2)$.

iv. Evaluate the total derivative of $z = 4x^3y + 7x^2y^3$ where $x = 4 + 4t^4$ and $y = 1 - 2t^2$, using chain rule.

Q4. Attempt any **ONE** question from the following: (08)

a) i. State and prove Taylor's Theorem for a real valued function of two variables.

ii. Let $Q(x,y) = Ax^2 + 2Bxy + Cy^2$ be a function of two variables and $\Delta = AC - B^2$. Then prove that

(1) if $\Delta > 0$ and $A > 0$ then $Q(x,y) > 0\ \forall\ (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$.

(2) if $\Delta > 0$ and $A < 0$ then $Q(x,y) < 0\ \forall\ (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$.

if $\Delta < 0$, then in every open ball around origin there exist points (x,y) such that $Q(x,y) < 0$ and there exist points (x,y) such that $Q(x,y) > 0$.

Q4. Attempt any **TWO** questions from the following: (12)

b) i. Given $z = f(x,y)$ where f has continuous partial derivatives of second order, $x = u + v,\ y = u - v$, show that $\frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$

ii. a) If $f(x,y,z) = xi + yj + zk$ then prove that the Jacobian matrix $Df(x,y,z)$ is the identity matrix of order 3.

b) Find all differentiable vector fields $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for which the Jacobian matrix $Df(x,y,z)$ is a diagonal matrix of form $diag\ (p(x), q(y), r(z))$ where p, q, r are given continuous functions.

iii. Find the critical points, saddle points and local extrema if any for the function $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x$.

iv. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find the distance.

Q5. Attempt any **FOUR** questions from the following: (20)

a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \lim_{(x,y)\to(0,0)} \frac{e^{-\left(\frac{1}{x^2+y^2}\right)}}{x^2+y^2}$ if $(x,y) \neq (0,0)$.

Define $f(0,0)$ so that f is continuous at origin.

$b)$ Find the real value of $\theta \in (0,1)$ if it exists, satisfying

$f(b) - f(a) = < \nabla f(a + \theta(b-a)), b - a >$

for the following function at the given points.

$f(x,y,z) = x^2 + y^2 + 2xz, \; a = (0,0,0), \; b = \left(1, \frac{1}{2}, \frac{1}{3}\right).$

$c)$ Find level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$ for the constants $K = 1,9$.

$d)$ Let $f(x,y) = x^2y^3 + 2y^5$, find $f_x, \; f_y, \; f_{xy}, \; f_{xx}, \; f_{yy}$.

$e)$ Using Taylor's formula find the quadratic approximation for the quantities $(0.99)^3 + (2.01)^3 - 6(0.99)(2.01)$.

$f)$ Find the Hessian matrix of $f: \mathbb{R}^3 \to \mathbb{R}$ given by

$f(x,y,z) = x^3 + y^3 + z^3 + 3xyz + 3x^2y + 3y^2x + 3z^2x + 3x^2z$ at (1,1,1).

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