(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if $\forall u, v \in \mathbb{R}^2, \forall \alpha, \beta \in \mathbb{R}$,

(a) $T(\alpha u+\beta v)=\alpha T(u)+\beta T(v)$ (b) $T(\alpha uv)=\alpha T(u)\cdot T(v)$

(c) $T(\alpha u+\beta v)=\alpha T(u)\cdot \beta T(v)$ (d) None of the above

$ii.$ If $T:U \to V$ is a linear transformation then

a $T(0) = 0$ (b) $T(-u) = -T(u), \forall\, u \in U$

c $T(u_1 - u_2) = T(u_1) - T(u_2)$ $,\forall u_1, u_2 \in U$ (d) All of the above

$iii.$ Which of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?

(a) $T(x,y)=(xy,y)$ (b) $T(x,y)=(x + 1, y + 1)$

(c) $T(x,y)=(x + y, x - y)$ (d) All the above

$iv.$ If $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{EA} = \begin{pmatrix} 0 & 3 \\ 2 & -2 \end{pmatrix}$, then $\mathbf{E}$ is given by

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$v.$ Which one of the following is NOT TRUE

(a) $Det(A^t) = Det\ A$ (b) $Det(A + B) = DetA + DetB$

(c) $Det(AB) = DetA\ DetB$ (d) $Det(A^{-1}) = (Det\ A)^{-1}$, when A is invertible.

$vi.$ $Det(e_2, 2e_1 + 3e_2, e_3)$ where e_1, e_2, e_3 are standard basis elements of $\mathbb{R}^3$ is

(a) -1 (b) 0

(c) 1 (d) -2

$vii.$ Let $A \in M_n(\mathbb{R})$ be an invertible matrix then det(Adj A) is

(a) $(det\ A)^n$ (b) $(det\ A)^{n-1}$

(c) $\det A^{-1}$ (d) None of these

$viii.$ Let V be a finite dimensional inner product space and W be a subspace of V.

Then $(W^{\perp})^{\perp}$ is equal to

(a) V (b) W

(c) $W^{\perp}$ (d) $V \backslash W$

$ix.$ For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, which of the following is **not** an inner product ?

(a) $\langle x, y\rangle = 2x_1y_1 + 3x_2y_2$ (b) $\langle x, y\rangle = x_1y_1 + x_2y_2$

(c) $\langle x, y\rangle = x_1y_1 - x_2y_2$ (d) $\langle x, y\rangle = x_1y_1 + 4x_2y_2$

x. If $\{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$ with Euclidean inner product, then for $x =$ $3v_1 + 7v_2$ and $y = 3v_1 - 7v_2$, $<x, y> =$ ____

(a) - 40 (b) 40

(c) 0 (d) None of these.

Q2. Attempt any **ONE** question from the following: (08)

$a)$ i. State and prove the Rank-Nullity Theorem.

ii. Let $A \in M_n(\mathbb{R})$. Prove that the system $AX=B$ of n non-homogenous linear equations in n unknowns has a unique solution if and only if rank$(A) = n$.

Q.2 Attempt any **TWO** questions from the following: (12)

$b)$ i. Show that F is non-singular where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $F(x,y,z) = (x+y-2z, x+2y+z, 2x+2y-3z)$.

ii. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find the basis for Ker T and Nullity T.

iii. Show that a n-dimensional real vector space is isomorphic to $\mathbb{R}^n$.

iv. Test for consistency and if consistent solve the system: $2x-3y+7z=5, 3x+y-3z=13, 2x+19y-47z=32$

Q3. Attempt any **ONE** question from the following: (08)

$a)$ i. Let $A \in M_n(\mathbb{R})$. Prove that A is invertible if and only if columns of A are linearly independent. Hence, prove that if $\det A = 0$ then columns of A are linearly dependent.

ii. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Show that

I) $\det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \det(v_1, \ldots, v_i + \propto v_j, \ldots, v_j, \ldots, v_n)$ for $1 \le i \ne j \le n$ and $\alpha \epsilon \mathbb{R}$

II) $\det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -\det(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n)$ for $1 \le i \ne j \le n$

Q3. Attempt any **TWO** questions from the following: (12)

$b)$ i. Let $A \in M_n(\mathbb{R})$. show that $A.adj(A) = \det A \,.\, I_n$, where I_n is the $n \times n$ identity matrix.

ii. Solve the following system of linear equations using Cramer's rule $2x - y + z = 1, x + 3y - 2z = 1, 4x - 3y + z = 0$

iii. For $A, B \in M_n(\mathbb{R})$, if A is invertible show that

a) $\det(A^{-1}) = (\det A)^{-1}$

b) $\det(ABA^{-1}) = \det B$

c) $\det(adj\ A) = (\det A)^{n-1}$

iv. Define adjoint of a matrix. Find $\boldsymbol{A^{-1}}$ for $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ using adjoint.

Q4. Attempt any **ONE** question from the following: (08)

a) i. Define inner product and inner product space over $\mathbb{R}$. Show that $(P_2[x], \langle \ , \ \rangle)$ is an inner product space over $\mathbb{R}$ where $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$ for $p(x) = a_0 + a_1x + a_2x^2,\ q(x) = b_0 + b_1x + b_2x^2$.

ii. Define orthogonal and orthonormal sets. Let $\{x_1, x_2, \cdots, x_n\}$ be an orthonormal basis of an inner product space V.
Let $x = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n$. Then prove the following:
(p) $\alpha_i = \langle x, x_i\rangle$ for $i = 1,2,..,n$
(q) $\| x \|^2 = \sum_{i=1}^{n}\langle x, x_i\rangle^2$

Q4. Attempt any **TWO** questions from the following: (12)

b) i. Define angle between two vectors in a real inner product space. Find angle between $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ with respect to the inner product $\langle A, B\rangle = tr(AB^t)$ on $M_2(\mathbb{R})$.

ii. Prove that an orthogonal set in a real inner product space V is linearly independent.

iii. Let W be a subspace of a real inner product space V. Define $W^\perp$, the orthogonal complement of W. Show that $W^\perp$ is a subspace of V.

iv. Apply Gram-Schmidt process to obtain orthogonal set corresponding to $\{(0,1,1), (1,-1,0), (2,0,1)\}$ in $\mathbb{R}^3$ with dot product.

Q5. Attempt any **FOUR** questions from the following: (20)

a) Prove that if $T : V \to V'$ is a linear transformation then T is injective if and only if $\ker T = \{0\}$.

b) Find the rank of $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$.

c) Use the following expression of determinant
$\det A = \sum_{\sigma\epsilon S_n} sgn\sigma \ a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} \ldots\ldots a_{n\,\sigma(n)}$
to find the determinant of the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

d) (I) Use determinant to check whether the homogeneous system
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -6 & 1 \\ 7 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has non-trivial solution. State the result used.

(II) Use determinant to find area of the parallelogram spanned by vectors.

$x = (5,6)$ and $y = (2,5)$. State the result used.

$e)$ Let V be an inner product space and $u, v \in V$. Let a, b be nonzero elements of $\mathbb{R}$ & $a \neq \pm b$. Prove that $\|au + bv\|=\|bu + av\|$ iff $\|u\| = \|v\|$

$f)$ Find distance between $f(x) = \cos x$ and $g(x) = \sin x$ in $C[-\pi, \pi]$ using $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$

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