VCD -- 15/10/19 - MATHS I- SYBSC - SEM III EXAM - 100 MARKS - 3HRS

N.B : 1) All questions are compulsory

2) Figures to right indicate full marks

Q.1 Choose correct alternative in each of the following (2 marks each)

1) The set $\{(x,y) \in \mathbb{R}^2 / 1 \le x + y < 4\}$ is -------in $\mathbb{R}^2$.

a) both open and closed b) open c) neither open nor closed d) none of these

2)The sequence $(x_m) \to a \ in \ \mathbb{R}^n$ iff ----------

a) $\|x_m - a\| \to 0 \ in \ \mathbb{R}$ b) $\|x_m - a\| \to 0 \ in \ \mathbb{R}^n$ c) $\|x_m - a\| \to 1 \ in \ \mathbb{R}^n$

d) none of these

3) The sequence $x_m = (m, m+1, m+2)$ is ----------

a) bounded but not convergent in $\mathbb{R}^3$ b) convergent but not bounded in $\mathbb{R}^3$

c) neither bounded nor convergent in $\mathbb{R}^3$ d) none of these

4) $f: \mathbb{R}^2 \to \mathbb{R}$ as $f(x,y) = 1$ $(x,y) = (0,0)$

$= -1$ $(x,y) \neq (0,0)$ then ---------

a) f is continuous but |f| is not continuous at (0,0)

b) f is not continuous but |f| is continuous at (0,0)

c) neither f nor |f| is continuous at (0,0)

d) none of these

5) If $Df(a)$ is -------------for a differentiable function $f: \mathbb{R}^n \to \mathbb{R}$

a) a linear transformation but not an element of $\mathbb{R}$

b)a real number c) not both linear transformation as well as an element of $\mathbb{R}$ d) none of these

6) For differentiable function $f: S \to \mathbb{R}$ at $\in S$, directional derivative of f at a in direction of u, $D_u f(a) = ------$

a) $\nabla f(a)(u)$ b) $\nabla f(a)$ c) 0 d) none of these

7) For a function : $\mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial x_i}(a)$ is ----------

a) directional derivative of f at a in direction of any nonzero vector u

b) directional derivative of f at a in direction of $(0,0,\ldots\ldots,1,\ldots\ldots 0) = e_i$

c) directional derivative of f at a in direction of $(1,0,\ldots\ldots,0,\ldots\ldots 0) = e_1$

d) none of these

8) let $f(x,y) = sinxy + \log(x+y)$ then taylor's polynomials of degree 2 about (1,0) is-----

a) $3x + 2y - x^2 - y^2 - 2$ b) $-x^2$ c) $3x + 2y - x^2$ d) none of these

9) For $g: \mathbb{R}^3 \to: \mathbb{R}^2$ $as\ g(u,v,w) = (uvw, u^2 + v^2 + w^2)$ $Jg(u,v,w) = ----$

a) $\begin{bmatrix} u & w \\ v & uv \end{bmatrix}$ b) $\begin{bmatrix} uw & uw & uv \\ 2u & 2u & 2w \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 0 & u \end{bmatrix}$ d) none of these

10) 10) If $z = f(x,y)$ $where\ x = rcos\theta, y = rsin\theta\ then$ -----

a) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0$ b) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2$ c) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

d) none of these

Q.2 a) Attempt any ONE question from the following.(8marks each)

1) Let the sequence $x_m = (x_1, x_2, \ldots\ldots, x_n) \in \mathbb{R}^n$ then prove that x_m is convergent in $\mathbb{R}^n$ iff each of the coordinate sequence x_i is convergent in $\mathbb{R}$

2)State and prove Mean Value theorem for scalar field.

b) Attempt any TWO question from the following.(6marks each)

1) Find the directional derivative of following function at indicated points and direction if exists.

$f: \mathbb{R}^2 \to \mathbb{R}$ as $f(x,y) = \frac{xy}{x^2+y^2}$ $(x,y) = (0,0)$

$= 0$ $(x,y) \neq (0,0)$

at $a = (0,0)$ in the direction of $u = (1,-1)$

2)Find the real value of $\theta \in (0,1)$ such that $f(a+v) - f(a) = D_u f(a + \theta v)||v||$ where u is unit vector in direction of v

$$f(x,y) = x^2 - xy \quad a = (-1,1), \quad v = (2,3)$$

3) Let S be a nonempty open subset of $\mathbb{R}^n$.Let $f, g: S \to \mathbb{R}$, $a = (a_1, a_2, \ldots\ldots, a_n) \in S$

If $\frac{\partial f}{\partial x_i}(a), \frac{\partial g}{\partial x_i}(a)$ exists for $i = 1,2,\ldots\ldots n$ then prove that

i) $\frac{\partial (f+g)}{\partial x_i}(a) = \frac{\partial f}{\partial x_i}(a) + \frac{\partial g}{\partial x_i}(a)$ ii) $\frac{\partial (\propto f)}{\partial x_i}(a) = \propto \frac{\partial f}{\partial x_i}(a)$

4) In the following find the partial derivatives of f at (0,0) if exists

i) $f(x,y)=\sqrt{|xy|}$ ii) $f(x,y)=x\sin\frac{1}{y}+y\sin\frac{1}{x}$ $xy\neq 0$

$=0$ otherwise

Q.3 a) Attempt any ONE question from the following.(8marks each)

1) State and prove Chain rule for scalar fields

2) Let S be a nonempty open subset of $\mathbb{R}^n$.Let $a\in S$ and suppose $\nabla f,\nabla g,\nabla(\propto f)$ exists at a.Then prove that i) $\nabla(\propto f)(a)=\propto\nabla f(a)$ where $\propto$ is real constant. ii) $\nabla\left(\frac{f}{g}\right)(a)=\frac{g(a)\nabla f(a)-f(a)\nabla g(a)}{(g(a))^2}$ provided $g(a)\neq 0$ and $g(x)\neq 0$ in neighbourhood of a

b) Attempt any TWO question from the following.(6marks each)

1) State and prove Euler's Theorem for function of three variables

2) State and prove Mean Value theorem for differentiable scalar fields

3)i)Use chain rule to find total derivative $f(x,y)=3x^3y^2+5x^2y^3$

$$x(t)=1-t^2, y(t)=1+t^2$$

ii) Find the level curve of following f for given k

$$f(x,y,z)=x^2+y^2+z^2 \qquad k=1$$

4)Prove that following functions are continuous but not differentiable at origin $f(x,y)=|xy|$

Q.4 a) Attempt any ONE question from the following.(8marks each)

1)State and prove the relation between total derivative and jacobian matrix of vector valued function

2) State and prove Mean Value Inequality for vector field f differentiable over nonempty subset S of $\mathbb{R}^n$.

b) Attempt any TWO question from the following.(6marks each)

1) i)Define the Jacobian matrix of a vector field at the given point

$$f(x,y)=(x\cos y, y\sin x) at \left(\frac{\pi}{4},\frac{\pi}{4}\right)$$

ii) Using chain rule find $\frac{\partial w}{\partial s},\frac{\partial w}{\partial t}$ $at\ s=1, t=2,\ w=xy+yz+zx$

$$x(s,t)=e^{st}, y(s,t)=t^2, z(s,t)=(s+t)^2\ at\ (s,t)=(1,-1)$$

2) Let S be a nonempty open subset of $\mathbb{R}^n$.Let $f:S\to\mathbb{R}^m$ be a vector field. If f is differentiable at $a\in S$ then it is continuous at $a\in S$. What about converse?

3) Let S be a nonempty open subset of $\mathbb{R}^2$. Let $f: S \to \mathbb{R}^m$ be a vector field. If f is differentiable at $a \in S$ then $\exists\, M > 0, \delta > 0$ such that $||x - a|| < \delta \Rightarrow ||f(x) - f(a)|| \leq M||x - a||$

4) Let S be a nonempty open subset of $\mathbb{R}^n$. Let $f: S \to \mathbb{R}^m$ be a vector field differentiable over S. Let $a, b \in S$ and the line segment joining a and b which is the set

$\{a + (b - a)t / t \in [0,1]\}$ lies in S then $f(b) - f(a) = \left(\int_0^1 J(f(a + (b - a)t)dt)\right).(b - a)$

$$= \left(\int_0^1 Df(a + (b - a)t)dt\right).(b - a)$$

Q.5 Attempt any FOUR question from the following.(5marks each)

1) Prove that every linear transformation T: $\mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$

2) i) Evaluate $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$ and $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ for following and check whether both are equal or not. $f(x, y) = \frac{2x}{x^2+y^2}$ $\quad (x, y) \neq (0,0)$

=0 otherwise

ii) In following find α so that $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (0,0)

$$f(x, y) = \frac{x^2y^2}{x^2 + y^2} \quad (x, y) \neq (0,0)$$

$= \alpha$ otherwise

3) i)Find total derivative of following function at indicated point

$f(x, y, z) = x^2 + y^2 + z^2$ at $a = (1,0,1)$

ii)Find directional derivative of following function at indicated point using gradient function.

$f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $a = (1,1,0), u = (1, -1,2)$

4) Prove that following functions are differentiable at origin.

$$f(x, y) = xy \sin \frac{1}{\sqrt{x^2 + y^2}} \quad xy \neq 0$$

= 0 otherwise

5) Use Lagranges multiplier method to find maximum and minimum values of given function subject to specified constraints. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 3z^2 = 6$

6)i) Define Hessian Matrix for $f: \mathbb{R}^n \to \mathbb{R}$ scalar field and find hessian matrix for

$f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^4 + 2xy^2z + y^2z$ at $(1,0,1)$

ii) Define the following terms for $f: S \to \mathbb{R}$ for nonempty open subset S of $\mathbb{R}^n$

a) stationary point of f b) absolute maximum at a of f c) saddle point of f