(3 Hours) [Total Marks : 100]

N.B. **1**. All questions are compulsory.

2. Figures to the right indicate marks for respective parts

Q.1 Choose correct alternative in each of the following: (20)

i. The set $S = \{(x,y) \in \mathbb{R}^2 \ / \ 1 \leq x^2 + y^2 \leq 2\}$ is

(a) An open set (b) Closed set

(c) Neither open nor closed (d) None of these

ii. Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & if\ (x,y)\neq(0,0) \\ 0 & if\ (x,y)=(0,0) \end{cases}$. Then,

(a) $\lim_{(x,y)\to(0,0)} f(x,y)$ exist. (b) f is continuous at $(0,0)$.

(c) $|f(x,y)| \leq \frac{1}{2}, \forall\ (x,y) \in \mathbb{R}^2$ (d) None of these

iii. Let $f(x,y) = |x| + |y|, for\ (x,y) \in \mathbb{R}^2$, then

(a) $f_x(0,0) = 0,\ f_y(0,0) = 0$ (b) $f_x(0,0)\ and\ f_y(0,0) do\ not\ exist.$

(c) $f_x(0,0) = 1,\ f_y(0,0) = 1$ (d) None of these

iv. Let A: Gradient of a scalar field is a scalar.

B: Every differentiable scalar field is continuous.

Then which of the following is true?

(a) A is true, B is false. (b) A is false, B is true.

(c) Both A & B are true. (d) Both A & B are false.

v. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded .Then

(a) f may or may not be continuous at all points (b) f is continuous at all points

(c) f is differentiable at all points (d) None of these

vi. The total derivative T_a of a scalar field is _________.

(a) a constant (b) a linear transformation

(c) a vector (d) a real number

vii. The linear approximation to $e^x cos(y+z)$ near the origin is

(a) independent of x. (b) independent of y

(c) independent of z (d) 1

viii. If $z = f(x, y)$ is differentiable and $x = g(u, v)$, $y = h(u, v)$ are also differentiable functions then $\frac{\partial z}{\partial u}$ is

(a) $\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ (b) $\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} - \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

(c) $\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ (d) $\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

ix. If $f(x, y) = x^2 - 2x + y^2$ then

(a) (1,0) is a critical point of f which is a local minima. (b) (1,0) is a critical point of f which is a local maxima.

(c) (1,0) is a saddle point of f. (d) None of these

x. Stationary point is a point where function $f(x, y)$ have

(a) $\frac{\partial f}{\partial x} = 0$. (b) $\frac{\partial f}{\partial y} = 0$

(c) Both (a) and (b). (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function and let $a \in \mathbb{R}^n$. Prove that f is continuous at a if and only if each $f_i: \mathbb{R}^n \to \mathbb{R}$ is continuous at a.

ii. State and prove mean value theorem for derivatives of scalar fields.

b) Attempt any TWO questions from the following: (12)

i. Using definition of limit, check if $\lim_{(x,y)\to(0,0)} f(x, y)$ exists, where

$$f(x, y) = \begin{cases} x \sin\frac{1}{y} & if\ y \neq 0 \\ 0 & if\ y = 0 \end{cases}.$$

ii. Define limit of a function $f : S \to \mathbb{R}$ where $S(\neq \phi) \subseteq \mathbb{R}^n$, at point a, and show that the limit of function of several variables is uniquely determined

iii. Define directional derivative of a scalar field $f(x, y, z) = \left(\frac{x}{y}\right)^z$, $y \neq 0$ at the point $a = (1,1,1)$, in the direction of $u = (2,1,-1)$ Find the directional derivative of the following functions at the indicated point in the direction (using the result $D_u f(a) = \nabla f(a).u$)

iv. If $\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$; $(x, y, z) \neq (0, 0, 0)$. Then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

Q.3 a) Attempt any ONE question from the following: (08)

i. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function and $a \in \mathbb{R}^n$. Define the total derivative $Df(a)$ in terms of a linear transformation and show that $Df(a)$ when exist, is uniquely defined.

ii. State and prove chain rule for the derivative of a scalar field.

b) Attempt any TWO questions from the following: (12)

i. Find total derivative as linear transformation T for the function $f(x, y, z) = x^3 + 3y^2 + 4z^2$ at point $a = (-1, -1, 1)$

ii. Find directional derivative of $f(x, y, z) = 3x - 5y + 2z$ at $(2,2,1)$ in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$.

iii. Find the equation of tangent plane and normal line to the surface $z + 1 = xe^y \cos z$ at $(1,1,0)$

iv. Show that, for each of the following functions, the second order mixed partial derivatives are equal.

1. $f(x, y) = x^4 + y^4 - xy^3$
2. $f(x, y) = \sqrt{xy}$

Q.4 a) Attempt any ONE question from the following: (08)

i. Define $Df(a)$, the total derivative at $a \in \mathbb{R}^n$ for a function $f: \mathbb{R}^n \to \mathbb{R}^m$ in terms of a linear transformation. Show that if f is differentiable at a then f is continuous at a. Is the converse true ? Explain.

ii. Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar field. Let $(a, b) \in S$ be a stationary point of f. Suppose $f(x, y)$ possesses continuous second order partial derivatives in some neighbourhood of (a, b). Let $A = f_{xx}(a, b), B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$ and $\Delta = AC - B^2$. Then prove that

(1) if $\Delta > 0, A > 0$, then f has local minimum at (a, b).

(2) if $\Delta > 0, A < 0$, then f has local maximum at (a, b).

(3) if $\Delta < 0$, then f has saddle point at (a, b).

if $\Delta = 0$, the test is inconclusive (show this by example).

b) Attempt any TWO questions from the following: (12)

i. Let U is open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^m$ is differentiable at $a \in U$. Show that $\exists\, M > 0, \delta > 0$ such that $\|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| \le M\|x - a\|$.

ii. Find the Taylors polynomial of degree 2 at $p = (1, \pi)$ for the function $f(x, y) = \cos(xy)$

iii. Find the critical points, saddle points and local extrema if any for the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

iv. Find the greatest area that a rectangle can have if the length of its diagonal is 2.

Q.5 Attempt any FOUR questions from the following: (20)

a) Show that for the following functions the limit does not exists

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6+y^2}$ (ii) $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^4+z^4}$

b) Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \begin{cases} 0 & if\ xy = 0 \\ 1 & if\ xy \neq 0 \end{cases}$

Show that f is not continuous at (0, 0) but both partial derivatives exist at (0,0)

c) Find the direction in which function $f(x, y) = 9x^3 + 5y^2$ increases most rapidly and the direction in which decreases most rapidly at point (2,1) .

d) Evaluate the total derivative of $z = 4x^3y + 7x^2y^3$ where $x = 4 + 4t^4$ and $y = 1 - 2t^2$, using chain rule.

e) Compute the matrices $Dg(1, 1), D(f(g(1, 1)))$ and $D(fog(1, 1))$ in each of the following and verify that

$D(f(g(1, 1)))Dg(1, 1) = D(fog(1, 1))$ where

$$f(u, v) = (e^{uv}, uv), \qquad g(x, y) = (x + y, x - y).$$

f) Find the Hessian matrix of $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 + z^2 + 3xy + 3yz + 3zx$ at (1,1,1).

****